Elliptic Partial Differential Equations

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Lecture notes based on the Elliptic Partial Differential Equations course lectured by Dr. N. Wickramasekera in Michaelmas term 2007 for Part III of the Cambridge Mathematical Tripos.

Contents

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1 Introduction

We begin by recalling some relevant definitions and results from calculus and measure theory that we will need throughout the course. We then give a brief explanation of the type of problems we will encounter in this course, and the methods we need to develop in order to solve them. As a concrete example, we conclude the introduction by discussing the minimal surface equation.

1.1 Definitions

- A **domain** $\Omega \subseteq \mathbb{R}^n$ is an open, connected set. For a differentiable function u, we write Du for the **gradient** of u, $Du = (D_1 u, \dots, D_n u)$ where $D_i u = \frac{\partial u}{\partial x_i}$.
- If Ω is a domain and $\mathcal{O}$ is an open set contained in Ω we write $\mathcal{O} \subset\subset \Omega$ to indicate that $\bar{\mathcal{O}} \subseteq \Omega$ and $\bar{\mathcal{O}}$ is compact.
- For a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$D^{(\alpha)} u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}},$$

  where $|\alpha| = \sum_{i=1}^n \alpha_i$. We set $\alpha! := \alpha_1! \dots \alpha_n!$.
- The **Laplacian** Δu is by definition $\sum_{i=1}^n D_{ii} u$, and the **divergence** $\operatorname{div}(u) = \sum_{i=1}^n D_i u$.
- $B_R(x)$ will denote the **open** ball of radius R about x and ω_n denotes the volume of $B_1(0) \subseteq \mathbb{R}^n$, that is,

$$\omega_n = \int_{B_1(0)} dx.$$

- A domain Ω has C^1 **boundary** $\partial\Omega$ if the following condition holds: for any $\xi \in \partial\Omega$ there exists $R > 0$ and a C^1 function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ such that after relabelling and reorientating the coordinates if necessary,
$$\Omega \cap B_R(\xi) = \{x \in B_R(\xi) \mid x_n > f(x_1, \dots, x_n)\}$$
(and so $\partial\Omega \cap B_R(\xi) = \{x \in B_R(\xi) \mid x_n = f(x_1, \dots, x_n)\}$). Likewise $\partial\Omega$ is C^k if f is C^k.

- The **outward pointing unit normal vector field** ν is then defined along $\partial\Omega$. Write $\nu = (\nu_1, \dots, \nu_n)$. The **unit normal** at any point $x_0 \in \Omega$ is $\nu(x_0)$. If $u \in C^1\left(\bar{\Omega}\right)$ then the **outward normal derivative** of u is $\frac{\partial u}{\partial \nu}$, which is by definition $\nu \cdot Du$. We write $dS\,(x)$ for the volume element of $\partial\Omega$, although often we will omit the dependence on the variable x, and just write dS.

1.2 Calculus results

- The **Divergence Theorem** states the following: let Ω be a domain with $\partial\Omega$ C^1. Let $u \in C^1\left(\bar{\Omega}\right)$. Then for $i = 1, \dots, n$,
$$\int_\Omega D_i u dx = \int_{\partial\Omega} u\nu_i dS(x).$$
Strictly the above formula is obtained from the Divergence Theorem by setting
$$\mathbf{u} = (0, \dots, 0, u, 0, \dots, 0),$$
and then the Divergence Theorem states that
$$\int_\Omega \operatorname{div}(\mathbf{u}) dx = \int_{\partial\Omega} \mathbf{u} \cdot \nu dS(x),$$
which immediately gives the stated result. We will mainly use the first formula, and refer to this as the 'Divergence Theorem'.

- From this we easily deduce **integration by parts**. Suppose $u, w \in C^1\left(\bar{\Omega}\right)$. Then for $i = 1, \dots, n$,
$$\int_\Omega D_i u w dx = -\int_\Omega u D_i w dx + \int_{\partial\Omega} uw\nu^i dS(x).$$
$$\geq$$
This follows immediately from (1) by applying it to uw, noting $D_i(uw) = uD_iw + wD_iu$.

- We will also need **Green's forumlas**. Suppose $u, w \in C^2(\Omega)$. Then we have
$$\int_\Omega \Delta u dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS(x), \tag{1}$$
$$\int_\Omega Du \cdot Dw dx = -\int_\Omega u\Delta w dx + \int_{\partial\Omega} \frac{\partial w}{\partial \nu} u dS(x), \tag{2}$$
$$\int_\Omega u\Delta w - w\Delta u dx = \int_{\partial\Omega} u\frac{\partial w}{\partial \nu} - w\frac{\partial u}{\partial \nu} dS(x). \tag{3}$$
The first follows from (2), by setting '$u' = D_iu$ and '$w' \equiv 1$, and then summing over i. The second follows from (2) by setting '$w' = D_iw$ and summing over i. Finally, the third follows by exchanging the roles of u and w in the second, and subtracting.

- The final result from calculus we need is the **Coarea formula**: if $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and $\int_{\mathbb{R}^n} f dx < \infty$ then for any $0 < R \leq \infty$,
$$\int_{B_R(x_0)} f dx = \int_0^R \left(\int_{\partial B_\rho(x_0)} f dS(x) \right) d\rho$$
and in particular
$$\frac{d}{dR}\left(\int_{B_R(x_0)} f dx \right) = \int_{\partial B_R(x_0)} f dS(x). \tag{4}$$

1.3 Meausure theory and integration

During the course we will at various stages quote the following four results from measure theory.

- **Egoroff's theorem** states that if $\Omega \subseteq \mathbb{R}^n$ is a measurable set with $|\Omega| < \infty$ and functions $\{f_k : \Omega \to \mathbb{R}\}$ are measurable such that
$$f_k \to f \quad \text{a.e. on } \Omega,$$
for some measurable function $f : \Omega \to \mathbb{R}$ then for any $\epsilon > 0$ there exists a measurable subset $E_\epsilon \subseteq \Omega$ such that $|\Omega \backslash E_\epsilon| \leq \epsilon$ and such that $f_k \to f$ uniformly on E_ϵ.

- The **monotone convergence theorem** states that if functions $\{f_k : \Omega \to \mathbb{R}\}$ are integrable with $f_k(x) \leq f_{k+1}(x)$ for all k and all $x \in \Omega$ then
$$\int_\Omega \lim_{k\to\infty} f_k = \lim_{k\to\infty} \int_\Omega f_k.$$

- **Lebesgue's dominated convergence theorem** states that if functions $\{f_k : \Omega \to \mathbb{R}\}$ are integrable and
$$f_k \to f \quad \text{a.e.},$$
and there exists a function $g \in L^1(\Omega)$ such that
$$|f_k| \leq g \quad \text{a.e.},$$
then
$$\int_\Omega \lim_{k\to\infty} f_k = \lim_{k\to\infty} \int_\Omega f_k.$$

- **Lebesgue's differentiation theorem** states that if $u : \Omega \to \mathbb{R}$ is locally summable then for a.e. $x \in \Omega$ we have
$$\frac{1}{\omega_n R^n} \int_{B_R(x)} u(y) \to u(x) \text{ as } R \downarrow 0.$$
In fact for a.e. $x \in \Omega$ we have
$$\frac{1}{\omega_n R^n} \int_{B_R(x)} |u(y) - u(x)| dy \to 0.$$
Such a point x is called a **Lebesgue point** of u.

Finally, we make the following definition. Let $\Omega \subseteq \mathbb{R}^n$ be an open domain. In addition to the standard spaces $L^p(\Omega)$ we define the spaces $L^p_{\text{loc}}(\Omega)$ consisting of the functions that are **locally** of class L^p; more precisely, $u : \Omega \to \mathbb{R}$ is in $L^p_{\text{loc}}(\Omega)$ if for every $\mathcal{O} \subset\subset \Omega$ we have $u \in L^p(\mathcal{O})$.

1.4 Variational Problems

In this course we will look at certain **variational problems**. Here is the general idea.

Problem: Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be smooth, where $\Omega \subseteq \mathbb{R}^n$ is a domain with smooth boundary,and consider the functional
$$\mathcal{F} : C^2(\Omega) \to \mathbb{R}$$
defined by (♠)
$$\mathcal{F}(u) := \int_\Omega F(x, u, Du) dx.$$
Suppose $g \in C^0(\partial\Omega)$. Does there exist a **minimizer** u of $\mathcal{F}$ that satisfies the **boundary condition** $u = g$ on $\partial\Omega$?

As we shall see, often minimizers do not exist in the function class we are considering, and we are forced to relax the regularity requirements on the domain of $\mathcal{F}$ in order to get a minimizer. Sometimes it will turn out however that although we could only prove existence of a minimizer in a certain class of functions possessing little regularity, once we know the minimizer exists we can then independently prove that the minimizer actually possesses rather more regularity.

Thus given a variational problem $\mathcal{F}$, three natural problems are:

1. Existence of minimizers in suitable classes of functions (eg. Sobolev spaces).
2. Regularity of the minimizer (often smoothness).
3. Uniqueness of the minimizer.

For some variational problems, the proofs of the three problems above are entirely separate. Sometimes one is a consequence of another: towards the end of the course we shall see a suprising example where to prove existence follows from existence, that is, we first prove that *if a solution exists it is unique*, and then use this to deduce that a solution *must exist*!

1.5 Euler-Lagrange equations

Consider again Problem ($\spadesuit$) above. Assume we have a $C^2(\Omega)$ minimizer u of $\mathcal{F}(\cdot)$ subject to $u = g$ on $\partial\Omega$. Let $\varphi \in C_c^1(\Omega)$, that is, let φ be a continuously differentiable function with compact support within Ω, and consider $u + s\varphi$ for $s \in (-\epsilon, \epsilon)$. These are **variations** of u. If

$$i(s) := \int_\Omega F(x, u + s\varphi, D(u + s\varphi))dx,$$

then by assumption i has a minimum at $s = 0$. This leads to the the **Euler-Lagrange** equations.

$$i'(0) = 0.$$

Explicitly,

$$\begin{aligned} 0 &= \frac{d}{ds}\Big|_{s=0} \int_\Omega F(x, u + s\varphi, D(u + s\varphi))dx \\ &= \int_\Omega \sum_{i=1}^n \big(F_{p_i}(x, u, Du)D_i\varphi + F_z(x, u, Du)\varphi\big)dx, \end{aligned}$$

where $F = F(x, z, p)$. Since φ has compact support, we may integrate by parts to obtain (using $\varphi = 0$ on $\partial\Omega$)

$$0 = \int_\Omega -\sum_{i=1}^n (D_i(F_{p_i}(x, u, Du)) + F_z(x, u, Du))\varphi dx.$$

Since this holds for all test functions φ, we conclude that u solves the nonlinear PDE in Ω,

$$\sum_{i=1}^n D_i(F_{p_i}(x, u, Du)) = F_z(x, u, Du). \tag{5}$$

We shall study the Euler-Lagrange equations in much greater depth in Chapter 9, after we have developed considerable technical machinery. Then we will be able to at least partially answer the three questions posed above. Let us however conclude this introductory example by looking at a concrete example of a variational problem.

1.6 The minimal surface equation

Here we consider $F(x, u, Du) = \sqrt{1 + |Du|^2}$. Then $\mathcal{F}(\cdot)$ is the **area functional**; if $u : \Omega \to \mathbb{R}$ and G_u is its graph then the area of G_u is

$$\mathcal{A}(u) = \int_\Omega \sqrt{1 + |Du|^2}dx = \mathcal{F}(u).$$

The associated Euler-Lagrange equation, called the **minimal surface equation** is

$$D_i\left(\frac{D_i u}{\sqrt{1+|Du|^2}}\right) = 0,$$

or

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0.$$

As written, this is the **divergence form** of the minimal surface equation. One can show that $\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)$ is n times the **mean curvature** of G_u; thus a minimal surface has zero mean curvature. We can also require obviously require $u = g$ on $\partial\Omega$, say.

We can explicitly carry out the differentiation to obtain the **non-divergence form**:

$$\frac{1}{1+|Du|^2}\left(\sqrt{1+|Du|^2}D_{ii}u - D_i u D_i(\sqrt{1+|Du|^2})\right),$$

which, using $D_i(\sqrt{1+|Du|^2}) = \frac{D_j u D_{ij} u}{\sqrt{1+|Du|^2}}$ we obtain

$$D_{ii}u - \frac{D_i u D_j u D_{ij} u}{1+|Du|^2} = 0,$$

or alternatively after summing

$$\Delta u - \frac{D_i u D_j u D_{ij} u}{1+|Du|^2} = 0$$

or

$$\left(\delta_{ij} - \frac{D_i u D_j u}{1+|Du|^2}\right) D_{ij} u = 0.$$

Let us conclude this introduction by quoting the following result.

1.7 Theorem (Bernstein)

If $u \in C^2(\mathbb{R}^n)$ is a solution to the minimal surface equation on all of $\mathbb{R}^n$, for $1 \le n \le 7$, then u is affine, u if $u(x) = ax + b$. This fails for $n \ge 8$.

2 Harmonic functions

2.1 Laplace's equation

As a second example we could take $F(x, u, Du) := \frac{1}{2}|Du|^2$, so $\mathcal{F}(\cdot)$ is the **Dirichlet Energy functional**,

$$\mathcal{E}(u) := \int_\Omega \frac{1}{2}|Du|^2.$$

We could also require $u = g$ on $\partial\Omega$, say. The associated Euler-Lagrange equation is just **Laplace's equation**:

$$\Delta u = 0 \text{ on } \Omega, u = g \text{ on } \partial\Omega.$$

2.2 Definitions

Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $u \in C^2(\Omega)$. We say that u is **harmonic** if $\Delta u = 0$ on Ω. We say u is **subharmonic** if $\Delta u \ge 0$ on Ω and we say that u is **superharmonic** if $\Delta u \le 0$ on Ω.

2.3 Theorem (Dirichlet's principle for Poisson's equation)

Suppose $\Omega \subseteq \mathbb{R}^n$ is a domain and $\partial\Omega$ is C^1. Let $f \in C^0(\Omega)$. Let

$$\mathcal{F}(w) := \int_\Omega \frac{1}{2}|Dw|^2 + wf$$

and let $g \in C^1(\partial\Omega)$. Set

$$\mathcal{C}_g := \{w \in C^2(\Omega) \cap C^1(\bar{\Omega}) \mid w = g \text{ on } \partial\Omega\}.$$

Then $u \in \mathcal{C}_g$ minimizes $\mathcal{F}(\cdot)$ if and only if $\Delta u = f$ on Ω, that is, u solves **Poisson's equation**.

◀ If u minimizes $\mathcal{F}(\cdot)$ then the calculation in Section 1.5 yields $\Delta u = f$ on Ω. Explicitly, if

$$i(s) := \int_\Omega \frac{1}{2}|D(u+s\varphi)|^2 + (u+s\varphi)f$$

for some $\varphi \in C_c^1(\Omega)$, and $s \in (-\epsilon, \epsilon)$, then we have

$$i'(0) = 0.$$

Now

$$i(s) = \int_\Omega \frac{1}{2}|Du|^2 + sDu \cdot D\varphi + \frac{s^2}{2}|D\varphi|^2 + (u+s\varphi)f dx,$$

and hence

$$0 = i'(0) = \int_\Omega Du \cdot D\varphi + \varphi f dx = \int_\Omega (-\Delta u + f)\varphi dx,$$

the latter equality using integration by parts, and the fact that $\varphi = 0$ on $\partial\Omega$. Since this holds for all $\varphi \in C_c^1(\Omega)$, we conclude $\Delta u = f$ on Ω.

For the converse, suppose $w \in \mathcal{C}_g$. Then

$$0 = \int_\Omega (-\Delta u + f)(u - w),$$

and integrating by parts gives

$$0 = \int_\Omega Du \cdot D(u-w) + f(u-w),$$

as there is no boundary term as $u - w = g - g = 0$ on $\partial\Omega$. Hence

$$\int_\Omega |Du|^2 + uf = \int_\Omega Du \cdot Dw + wf,$$

and using $|Du \cdot Dw| \leq |Du||Dw| \leq \frac{1}{2}|Du|^2 + \frac{1}{2}|Dw|^2$ we obtain

$$\int_\Omega Du \cdot Dw + wf \leq \int_\Omega \frac{1}{2}|Du|^2 + \int_\Omega \frac{1}{2}|Dw|^2 + wf$$

and hence by rearranging $\mathcal{F}(u) \leq \mathcal{F}(w)$. ▶

2.4 Theorem (Mean-value properties)

Let $\Omega \subseteq \mathbb{R}^n$ be open, and $\overline{B_R(x)} \subseteq \Omega$. Suppose $u \in C^2(\Omega)$ is subharmonic in Ω. Then

1. $u(x) \leq \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(x)} u$.
2. $u(x) \leq \frac{1}{\omega_n R^n} \int_{B_R(x)} u$.

If u is superharmonic then the inequalities are reversed, and if u is harmonic then we have equality. In the future we shall state results only for subharmonic functions; as always to obtain the corresponding result for superharmonic functions reverse the inequalities, and for harmonic functions we then have equality.

◀ We note that 1 holds in the limit as $R \to 0$ with equality, so we need only show that the right hand side is monotone increasing as R decreases.

Now if
$$\phi(R) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(y)} u(x) dS(x) = \frac{1}{n\omega_n} \int_{S^{n-1}} u(y+Rz) dS(z),$$
then
$$\begin{aligned}\phi'(R) &= \frac{1}{n\omega_n} \int_{S^{n-1}} Du(y+Rz)\cdot z dS(z) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(y)} Du \cdot \frac{x-y}{R} dS(x)\\ &= \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(y)} \frac{\partial u}{\partial \nu}(x) dS(x) = \frac{1}{n\omega_n R^{n-1}} \int_{B_R(y)} \Delta u(x) dx \geq 0.\end{aligned}$$
the last equality coming from (1), the first of Greens' inequalities.

This proves 1 and then 2 follows by integrating over the ball of radius R using the coarea formula (4):
$$\int_{B_R(x)} u = \int_0^R \left(\int_{\partial B_t(x)} u dS \right) dt = u(x) \int_0^R n\omega_n t^{n-1} dt = \omega_n R^n u(x). \quad ▶$$

In fact, the mean value properties characterise harmonic functions. See Proposition 2.15 below.

2.5 Proposition

Let $\Omega \subseteq \mathbb{R}^n$ be a domain with a C^1 boundary. Suppose $u \in C^2(\Omega) \cap C^1\left(\bar{\Omega}\right)$ satisfies $\Delta u = f$ on Ω and $u = g$ on $\partial\Omega$, where $f \in C^0(\Omega)$ and $g \in C^1(\partial\Omega)$. Then u is unique.

◀ Suppose u_1, u_2 are two solutions. Then if $v = u_1 - u_2$, then $\Delta v = 0$ on Ω and $v = 0$ on $\partial\Omega$. But then
$$0 = -\int_\Omega v\Delta v = \int_\Omega |Dv|^2,$$
and hence $Dv \equiv 0$. Since $v = 0$ on $\partial\Omega$, we conclude $v = 0$ on all of Ω. ▶

2.6 Corollary (Strong maximum principle for harmonic functions)

Let $\Omega \subseteq \mathbb{R}^n$ be a domain, $u \in C^2(\Omega)$ subharmonic. Then u does not attain a maximum in Ω unless u is constant.

◀ Suppose u attains a maximum M at $y \in \Omega$. Let $\Sigma := \{x \in \Omega \mid u(x) = M\}$. Then Σ is non-empty and closed (since u is continuous). Consider $v := M - u$. Then $v \geq 0$ and v is superharmonic. If $x \in \Sigma$, then for suitably small $R > 0$, by (2.3.2) we have $\overline{B_R(x)} \subseteq \Omega$ and
$$v(x) \geq \frac{1}{\omega_n R^n} \int_{B_R(x)} v.$$

But by assumption $x \in \Sigma$ implies the left hand side is zero, and the right hand side is non-negative as v is non-negative. Thus $v \equiv 0$ in $B_R(x)$ and thus Σ is open. Since Ω is connected, $\Sigma = \Omega$ as was required. ▶

2.7 Corollary (Weak maximum principle for harmonic functions)

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $u \in C^2(\Omega) \cap C^0\left(\bar{\Omega}\right)$ subharmonic. Then

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

◀ Note that the maximum is certainly attained somewhere, as $\bar{\Omega}$ is compact and u is continuous. By the strong maximum principle this maximum is attained on $\partial\Omega$. ▶

Both the strong and weak maximum prinicples hold for a much wider class of functions than harmonic functions. In general, one proves the weak maximum principle (the **global** result) first, and then the strong maximum prinicple (the **local** result). For harmonic functions though, Theorem 2.4 makes it is possible to go the other way round for a quick, simple proof.

2.8 Corollary

Let $\Omega \subseteq \mathbb{R}^n$ be a domain with a C^1 boundary. Suppose $u \in C^2(\Omega) \cap C^0\left(\bar{\Omega}\right)$ satisfies $\Delta u = f$ on Ω and $u = g$ on $\partial\Omega$, where $f \in C^0(\Omega)$ and $g \in C^0(\partial\Omega)$. Then u is unique.

Note that this is an improvement over Proposition 2.5, as we now only need u and g to be continuous at the boundary, not differentiable.

◀ The proof is immediate; using the notation of Proposition 2.5, we have $\Delta v = 0$ and $v = 0$ on $\partial\Omega$, whence by the weak maximum priniple:

$$\max_{\bar{\Omega}} w = \max_{\partial\Omega} w = 0 = \min_{\partial\Omega} w = \min_{\bar{\Omega}} w. \quad ▶$$

2.9 Theorem (Harnak inequalities)

Let $\Omega \subseteq \mathbb{R}^n$ be a domain, and $\mathcal{O} \subset\subset \Omega$ a subdomain. Then there exists a constant $C = C(n, \Omega, \mathcal{O}) > 0$ such that if $u \in C^2(\Omega)$ is a non-negative harmonic function then

$$\sup_{\mathcal{O}} u \leq C \inf_{\mathcal{O}} u.$$

Equivalently, given any $x_1, x_2 \in \mathcal{O}$ we have

$$\frac{1}{C} u(x_2) \leq u(x_1) \leq C u(x_2).$$

◀ Suppose $\overline{B_{4R}(y)} \subseteq \Omega$, and let $x_1, x_2 \in B_R(y)$. Then by Theorem 2.4.2,

$$u(x_1) = \frac{1}{\omega_n R^n} \int_{B_R(x_1)} u \leq \frac{1}{\omega_n R^n} \int_{B_{2R}(y)} u,$$

as u is non-negative and $B_R(x_1) \subseteq B_{2R}(y)$. Similarly,

$$u(x_2) = \frac{1}{\omega_n (3R)^n} \int_{B_{3R}(x_2)} u \geq \frac{1}{\omega_n (3R)^n} \int_{B_{2R}(y)} u,$$

and hence $u(x_1) \leq 3^n u(x_2)$.

Now let x_1 and x_2 be the points in $\bar{\mathcal{O}}$ be points where u attains its maximum and minimum respectively. Choose a path Γ from x_1 to x_2 contained in $\mathcal{O}$. Now choose $R > 0$ such that $4R < \operatorname{dist}(\Gamma, \partial\Omega)$. By compactness, we can cover Γ by $N = N(\Omega, \mathcal{O}) < \infty$ balls of radius $4R$, and then by the above we obtain $u(x_1) \leq 3^{nN} u(x_2)$. Thus we may take $C = C(n, \Omega, \mathcal{O}) = 3^{nN(\Omega, \Omega_0)}$. ▶

2.10 Definition

Let $\Omega \subseteq \mathbb{R}^n$ be a domain. A function $u \in L^1(\Omega)$ is **weakly harmonic** in Ω if

$$\int_\Omega u\Delta\varphi = 0$$

for all $\varphi \in C_c^2(\Omega)$. Note that if $u \in C^2(\Omega)$ is (classically) harmonic then u is weakly harmonic: by Greens' Theorem (2) applied twice we obtain (since the boundary terms vanish as φ is compactly supported)

$$0 = \int_\Omega u\Delta\varphi = -\int_\Omega Du \cdot D\varphi = \int_\Omega \Delta u\varphi,$$

and since this holds for all $\varphi \in C_c^2(\Omega)$, we conclude u is harmonic in Ω.

We can also define weakly subharmonic and weakly superharmonic in the same way.

2.11 Definition

Define $\eta \in C^\infty(\mathbb{R}^n)$ by

$$\eta(x) = \begin{cases} C\exp\left(\frac{1}{|x^2|-1}\right) & |x| < 1 \\ 0 & |x| \geq 1, \end{cases}$$

where C is selected such that $\int_{\mathbb{R}^n} \eta = 1$. We call η the **standard mollifier**. Given $\sigma > 0$, define η_σ by $\eta_\sigma(x) := \sigma^{-n}\eta(x/\sigma)$. Then the functions η_σ are smooth, $\int_{\mathbb{R}^n} \eta_\sigma = 1$ and η_σ is supported in the ball $B_\sigma(0)$. Note also that η_σ is a radial function.

2.12 Theorem (Weyl's Lemma)

Let $\Omega \subseteq \mathbb{R}^n$ be a domain. If $u \in L^1(\Omega)$ is weakly harmonic in Ω, then there exists a $\bar{u} \in C^\infty(\Omega)$ such that $\bar{u}$ is classically harmonic and $u = \bar{u}$ for a.e. $x \in \Omega$.

It will take some time to prove this theorem. The same result is **not** true for weakly subharmonic or weakly superharmonic functions.

◄ We proceed in several stages.

Step 1: The first step is to **mollify** u. Given $\sigma > 0$, define $\Omega_\sigma := \{x \in \Omega \mid d(x, \partial\Omega) > \sigma\}$. Now define $u_\sigma : \Omega_\sigma \to \mathbb{R}$ by

$$u_\sigma(x) = (\eta_\sigma * u)(x) = \int_{\mathbb{R}^n} \eta_\sigma(x-y)u(y)dy = \int_{B_\sigma(x)} \eta_\sigma(x-y)u(y)dy.$$

We call u_σ the **σth mollification of** u. We claim that $u_\sigma \in C^\infty(\Omega_\sigma)$.

Fix $x \in \Omega_\sigma$, and $1 \leq i \leq n$. Then if h is chosen small enough such that $x + he_i \in \Omega_\sigma$, we have

$$\frac{u_\sigma(x+he_i) - u_\sigma(x)}{h} = \frac{1}{\sigma^n}\int_{B_\sigma(x)} \frac{1}{h}\left(\eta\left(\frac{x+he_i-y}{\sigma}\right) - \eta\left(\frac{x-y}{\sigma}\right)\right)u(y)dy.$$

Since

$$D_i\eta_\sigma(x) = \sigma^{-n+1}D_i\eta\left(\frac{x}{\sigma}\right),$$

we have

$$\frac{1}{h}\left(\eta\left(\frac{x+he_i-y}{\sigma}\right) - \eta\left(\frac{x-y}{\sigma}\right)\right) \to \frac{1}{\sigma}D_i\eta\left(\frac{x-y}{\sigma}\right)$$

uniformly on $B_\sigma(x)$, and hence $D_iu_\sigma(x)$ exists and for $x \in \Omega_\sigma$,

$$D_iu_\sigma(x) = \int_{B_\sigma(x)} D_i\eta_\sigma(x-y)u(y)dy.$$

A similar argument shows that $D^{(\alpha)}u_\sigma(x)$ exists and for $x \in \Omega_\sigma$,

$$D^{(\alpha)}u_\sigma(x) = \int_{B_\sigma(x)} D^{(\alpha)}\eta_\sigma(x-y)u(y)dy$$

for any multiindex α. Thus $u_\sigma \in C^\infty(\Omega_\sigma)$.

Step 2: We now claim that $u_\sigma \to u$ for a.e. $x \in \Omega_\sigma$. Fix such a point $x \in \Omega_\sigma$. Then using the fact that η_σ has unit intgeral,

$$\begin{aligned} |u_\sigma(x) - u(x)| &= \left| \int_{B_\sigma(x)} \eta_\sigma(x-y)(u(y)-u(x))dy \right| \\ &\leq \frac{1}{\sigma^n} \int_{B_\sigma(x)} \eta\left(\frac{x-y}{\sigma}\right)|u(y)-u(x)|dy \\ &\leq \frac{C}{\sigma^n} \int_{B_\sigma(x)} |u(y)-u(x)| \to 0, \end{aligned}$$

by Lebesgue's Differentiation Theorem.

Step 3: Next, we claim that u_σ is classically harmonic in Ω_σ. Write Δ_x to indicate that the differentiation is with respect to x in the Laplacian. Then

$$\begin{aligned} \Delta_x u_\sigma(x) &= \Delta_x \left(\int_{B_\sigma(x)} \eta_\sigma(x-y)u(y)dy \right) \\ &= \int_{B_\sigma(x)} \Delta_x \left(\eta_\sigma(x-y)\right) u(y)dy, \end{aligned}$$

as η_σ is smooth. But by the chain rule,

$$\Delta_x\left(\eta_\sigma(x-y)\right) = \Delta_y\left(\eta_\sigma(x-y)\right),$$

as the (-1)'s cancel. But then

$$\int_{B_\sigma(x)} \Delta_y\left(\eta_\sigma(x-y)\right)u(y)dy = 0$$

since u is weakly harmonic and $f(y) := \eta_\sigma(x-y) \in C_c^2(\Omega)$.

Step 4: The next thing to prove are the following two statements about mollification. Let $\sigma, \tau > 0$. Define $(u_\sigma)_\tau(x) = \eta_\tau * u_\sigma$ for $\tau > 0$, so $(u_\sigma)_\tau$ is defined in $\Omega_{\sigma+\tau}$. Similarly we define $(u_\tau)_\sigma$. We claim for all $x \in \Omega_{\sigma+\tau}$:

1. $(u_\sigma)_\tau(x) = u_\sigma(x)$,
2. $(u_\sigma)_\tau(x) = (u_\tau)_\sigma(x)$.

Observe for any $x \in \Omega_{\sigma+\tau}$,

$$(u_\sigma)_\tau(x) = \int_{B_\tau(x)} \eta_\tau(x-y)u_\sigma(y)dy = \frac{1}{\tau^n}\int_{B_\tau(x)} \eta\left(\frac{x-y}{\tau}\right)u_\sigma(y)dy,$$

and by the coarea formula (4) we have

$$(u_\sigma)_\tau(x) = \frac{1}{\tau^{n-1}}\int_0^1 \int_{\partial B_{\tau\rho}(x)} \eta\left(\frac{x-y}{\tau}\right)u_\sigma(y)dyd\rho.$$

Now recall that η is **radial**, and thus $\eta(z) = \eta(|z|)$, and since on $\partial B_{\tau\rho}(x)$, we have

$$\left|\frac{x-y}{\tau}\right| = \rho,$$

we can write

$$(u_\sigma)_\tau(x) = \int_0^1 n\omega_n\rho^{n-1}\eta(\rho)\left(\frac{1}{n\omega_n(\tau\rho)^{n-1}}\int_{\partial B_{\tau\rho}(x)} u_\sigma(y)dy\right)d\rho,$$

and then applying Theorem (2.4).(1) to the harmonic function u_σ we obtain

$$(u_\sigma)_\tau(x) = u_\sigma(x)\int_0^1 n\omega_n\rho^{n-1}\eta(\rho)d\rho.$$

Finally,

$$\int_0^1 n\omega_n\rho^{n-1}\eta(\rho)d\rho = \int_0^1 \eta(\rho)\left(\int_{\partial B_\rho(0)} dS\right)d\rho = \int_{B_1(0)} \eta(|y|)dy = 1,$$

and thus we conclude $(u_\sigma)_\tau(x) = u_\sigma(x)$.

To prove the second statement, let $x \in \Omega_{\sigma+\tau}$ and observe we may take all our integrals to be over $\Omega_{\sigma+\tau}$. We have

$$\begin{aligned}(u_\sigma)_\tau(x) &= (\eta_\tau * u_\sigma)(x)\\ &= \int_{\Omega_{\sigma+\tau}} \eta_\tau(x-y)u_\sigma(y)dy\\ &= \int_{\Omega_{\sigma+\tau}} \eta_\tau(x-y)\int_{\Omega_{\sigma+\tau}} \eta_\sigma(y-z)u(z)dzdy.\end{aligned}$$

Now set $w = x - y + z$. Then

$$(u_\sigma)_\tau(x) = \int_{\Omega_{\sigma+\tau}} \eta_\tau(w-z)\int_{\Omega_{\sigma+\tau}} \eta_\sigma(x-w)u(z)dwdz,$$

which upon exchanging the order of integration (which is valid, as η_σ and η_τ are smooth and u integrable) is equal to $(u_\tau)_\sigma(x)$.

Step 5: We can now complete the proof. Fix some $\tau > 0$. We have shown that for a.e. $x \in \Omega_{\sigma+\tau}$ we have $(u_\tau)_\sigma(x) = (u_\sigma)_\tau(x) = u_\sigma(x)$. Now let $\sigma \to 0$. Thus for a.e. $x \in \Omega_\tau$, we have $u_\tau(x) = u(x)$, with u_τ smooth and classically harmonic. But τ was arbitrary; it follows there exists a smooth harmonic function $\bar{u}$ defined on all of Ω such that for a.e. $x \in \Omega$, $u(x) = \bar{u}(x)$. This completes the proof. ▶

From now on, when we talk about harmonic functions we may automatically assume that they are smooth.

2.13 Proposition (gradient estimates for harmonic functions)

Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, $u \in C^\infty(\Omega)$ be harmonic in Ω. Then:

1. For any $x \in \Omega$ and any $\rho > 0$ such that $B_\rho(x) \subseteq \Omega$, we have
$$|Du(x)| \le \frac{n}{\rho}\sup_{B_\rho(x)}|u|.$$
2. For any $x \in \Omega$ and any $R > 0$ such that $B_{2R}(x) \subseteq \Omega$, we have
$$\sup_{B_R(x)}|Du| \le \frac{n}{R}\sup_{B_{2R}(x)}|u|.$$
3. For any $x \in \Omega$ it holds that
$$|Du(x)| \le \frac{n}{\operatorname{dist}(x,\partial\Omega)}\sup_\Omega|u|.$$

◀ By differentiating Laplace's equation, we see that $D_i u$ is harmonic in Ω. If $x \in \Omega$ and $\rho > 0$ is such that $B_\rho(x) \subseteq \Omega$ then Theorem 2.4.1 gives

$$D_i u(x) = \frac{1}{\omega_n \rho^n} \int_{B_\rho(x)} D_i u,$$

and then the Divergence Theorem gives

$$D_i u(x) = \frac{1}{\omega_n \rho^n} \int_{\partial B_\rho(x)} u \nu_i dS \leq \frac{n\omega_n}{\omega_n \rho^n} \rho^{n-1} \sup_{\partial B_\rho(0)} |u|.$$

Then Corollary 2.7) then completes the proof of 1.

To prove 2, we note that the supremum of the continuous function $|Du(\cdot)|$ attains its maximum on the compact set $\overline{B_R(x)}$, and hence

$$\sup_{B_R(x)} |Du| = |Du(x_0)|$$

for some $x_0 \in \overline{B_R(x)}$. Hence

$$\begin{aligned} \sup_{B_R(x)} |Du| &= |Du(x_0)| \\ &\leq \frac{n}{R} \sup_{B_R(x_0)} |u| \\ &\leq \frac{n}{R} \sup_{B_{2R}(x)} |u|. \end{aligned}$$

Finally, 3 is an easy consequence of 2. ▶

2.14 Corollary (Liouville's Theorem)

If $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic on all of $\mathbb{R}^n$ and bounded then u is constant.

◀ For any $x \in \mathbb{R}^n$ and $1 \leq i \leq n$, letting $\rho \to \infty$ in Proposition 2.13.1 shows $Du(x) = 0$. We conclude $Du \equiv 0$ and thus u is constant. ▶

2.15 Proposition

1. If $u \in C^2(\Omega)$ satisfies the mean value property in Ω, that is $u(x) = (\omega_n R^n)^{-1} \int_{B_R(x)} u$ whenever $\overline{B_R(x)} \subseteq \Omega$ then u is harmonic in Ω.
2. If $u \in L^1(\Omega)$ and for a.e. $x \in \Omega$ and a.e. $R \in (0, \operatorname{dist}(x, \partial\Omega))$ we have $u(x) = (\omega_n R^n)^{-1} \int_{B_R(x)} u$ whenever $\overline{B_R(x)} \subseteq \Omega$ then u is harmonic in Ω.

◀ The case $u \in C^2(\Omega)$ is easy; if $\Delta u(x) \neq 0$ for some $x \in \Omega$, then there exists $R > 0$ such that $\Delta u \neq 0$ in $B_R(x)$, with $\overline{B_R(x)} \subseteq \Omega$. This violates the mean value property. If now u is only in $L^1(\Omega)$, then we observe that if

$$u_R(x) = (\eta_R * u)(x) = \int_{\mathbb{R}^n} \eta_R(x-y) u(y) dy = \int_{B_R(x)} \eta_R(x-y) u(y) dy,$$

then u_R is smooth (see Step 1 in the proof of Theorem 2.12) and if $\overline{B_R(x)} \subseteq \Omega$ then

$$\begin{aligned} u_R(x) &= \frac{1}{R^n}\int_0^R \int_{\partial B_\rho(x)} \eta\left(\frac{\rho}{R}\right) u(y)dS(y)d\rho \\ &= \frac{1}{R^d}\int_0^R \eta\left(\frac{\rho}{R}\right) n\omega_n\rho^{n-1}\left(\frac{1}{n\omega_n\rho^{n-1}}\int_{B_\rho(x)} u(y)dS(y)\right)d\rho \\ &= u(x)\int_0^1 \eta(z)n\omega_n z^{n-1}dz \\ &= u(x)\int_{\partial B_1(0)} \eta(|y|)dy \\ &= u(x). \end{aligned}$$

Hence u is infinitely differentiable a.e.. Moreover since a.e. for $\overline{B_\rho(x)} \subseteq \Omega$ we then have

$$\begin{aligned} \int_{B_\rho(x)} \Delta u(y)dy &= n\omega_n\rho^{n-1}\frac{\partial}{\partial\rho}\left(\frac{1}{n\omega_n\rho^{n-1}}\int_{\partial B_\rho(x)} u(y)dy\right) \\ &= n\omega_n\rho^{n-1}\frac{\partial}{\partial\rho}(u(x)) \\ &= 0, \end{aligned}$$

by the mean value property, it follows u is a.e. harmonic, too. ▶

2.16 Corollary

The limit of a uniformly convergent sequence of harmonic functions is harmonic.

◀ One simply notes that the uniform limit of functions satisfying the mean value property also satisfies the mean value property (the integral commutes with the limit) and applies the previous Proposition. ▶

2.17 Fundamental solutions

Laplace's equation is rotation invariant and thus it is desirable to look for **radial** solutions to Laplace's equation, that is, we seek harmonic functions $u : \mathbb{R}^n \to \mathbb{R}$ such that $u(x) = \Gamma(r)$ for some function Γ, where $r = |x|$. Now

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}(x_1^2 + \cdots + x_n^2)^{-\frac{1}{2}} \cdot 2x_i = \frac{x_i}{r}$$

and thus $D_i u = \Gamma'(r)\frac{x_i}{r}$, and

$$\begin{aligned} D_{ii}u &= \frac{\partial}{\partial x_i}\left(\Gamma'(r)\frac{x_i}{r}\right) \\ &= \frac{x_i}{r}\frac{\partial}{\partial x_i}(\Gamma'(r)) + \frac{1}{r}\Gamma'(r) \\ &= \frac{x_i}{r}\frac{d}{dr}\left(\frac{\partial r}{\partial x_i}\cdot\Gamma'(r)\right) + \frac{1}{r}\Gamma'(r) \\ &= \frac{x_i}{r}\left(\frac{d}{dr}\left(\frac{x_i}{r}\right)\Gamma'(r) + \frac{x_i}{r}\Gamma''(r)\right) \\ &= \frac{x_i^2}{r^2}\Gamma''(r) + \left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)\Gamma'(t). \end{aligned}$$

Summing over i gives the ODE

$$\Gamma'' + \frac{n-1}{r}\Gamma' = 0.$$

If $\Gamma' \neq 0$ we observe

$$\log(\Gamma')' = \frac{\Gamma''}{\Gamma'} = \frac{1-n}{r},$$

and hence $\Gamma'(r) = Cr^{1-n}$ for some constant C. Consequently for $r > 0$ we have

$$\Gamma(r) = \begin{cases} C_1 \log r + C_2 & n = 2 \\ C_1 r^{2-n} + C_2 & n \geq 3. \end{cases}$$

We thus define the **fundamental solution to Laplace's equation** to be (for $x \neq 0$)

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi} \log |x| & n = 2 \\ \frac{1}{n(n-2)\omega_n} |x|^{2-n} & n \geq 3. \end{cases}$$

We often abuse notation and write $\Gamma(|x|)$. Then for $x \neq y$, define $\Gamma(x, y) = \Gamma(x - y)$. The choice of constants will become clear soon. $\Gamma(x, y)$ is harmonic when $x \neq y$; it has a **singularity** at $x = y$. Observe that although $\Gamma(x, y) \to -\infty$ as $x \to y$, Γ remains of class L^1, that is

$$\int_{B_R(y)} |\Gamma(x, y)| dx < \infty \text{ for all } R > 0.$$

2.18 Theorem (Green's representation formula)

Let Ω be a C^1 domain and $u \in C^2\left(\bar{\Omega}\right)$. Then for $y \in \Omega$,

$$u(y) = \int_{\partial\Omega} \left(u(x) \frac{\partial \Gamma}{\partial \nu_x}(x, y) - \Gamma(x, y) \frac{\partial u}{\partial \nu}(x) \right) dS(x) + \int_\Omega \Gamma(x, y) \Delta u(x) dx,$$

where $\partial/\partial\nu_x$ indicated that the deriviative is to be taken in the direction of the interior normal with respect to the variable x.

◀ Choose $\rho > 0$ such that $\overline{B_\rho(y)} \subseteq \Omega$. We apply Green's formula (3) with $w(x) = \Gamma(x, y)$ and integrate over $\Omega \backslash B_\rho(y)$. Noting that Γ is harmonic in $\Omega \backslash B_\rho(y)$, we have

$$\begin{aligned} \int_{\Omega \backslash B_\rho(y)} \Gamma(x, y) \Delta u(x) dx &= \int_{\partial\Omega} \left(\Gamma(x, y) \frac{\partial u}{\partial \nu}(x) - u(x) \frac{\partial \Gamma}{\partial \nu_x}(x, y) \right) dS(x) \\ &\quad + \int_{\partial B_\rho(y)} \left(\Gamma(x, y) \frac{\partial u}{\partial \nu}(x) - u(x) \frac{\partial \Gamma}{\partial \nu_x}(x, y) \right) dS(x). \end{aligned}$$

Note that in the secondary boundary integral, ν denotes the exterior normal of $\Omega \backslash B_\rho(y)$ and hence the interior normal of $B_\rho(y)$. Since $u \in C^2\left(\bar{\Omega}\right)$, Δu is bounded. Since Γ is integrable, the left-hand side tends to

$$\int_\Omega \Gamma(x, y) \Delta(u) dx.$$

On $\partial B_\rho(y)$, we have $\Gamma(x, y) = \Gamma(\rho)$. Thus we have

$$\begin{aligned} \left| \int_{\partial B_\rho(y)} \Gamma(x, y) \frac{\partial u}{\partial \nu}(x) dS(x) \right| &\leq \Gamma(\rho) \int_{\partial B_\rho(y)} |Du(x) \cdot \nu(x)| dS(x) \\ &\leq \Gamma(\rho) n \omega_n \rho^{n-1} \sup_{B_\rho(y)} |Du|. \end{aligned}$$

Thus if $n = 2$, we have

$$\left| \int_{\partial B_\rho(y)} \Gamma(x, y) \frac{\partial u}{\partial \nu}(x) dS(x) \right| \leq C\rho \log \rho \to 0$$

as $\rho \to 0$, and if $n \geq 3$ we have

$$\left| \int_{\partial B_\rho(y)} \Gamma(x,y) \frac{\partial u}{\partial \nu}(x) dS(x) \right| \leq C\rho \to 0$$

as $\rho \to 0$.

Next, we observe

$$\frac{\partial \Gamma}{\partial \nu_x}(x,y) = D\Gamma(x,y) \cdot \nu(x) = D\Gamma(x,y) \cdot \frac{x-y}{\rho},$$

which for $n = 2$ gives

$$\begin{aligned} \frac{\partial \Gamma}{\partial \nu_x}(x,y) &= \frac{1}{2\pi} D(\log|x-y|) \cdot \frac{x-y}{\rho} \\ &= \frac{1}{2\pi} \frac{(x-y)\cdot(x-y)}{\rho^3}, \end{aligned}$$

since $\frac{\partial}{\partial x_i}(\log|x-y|) = \frac{x_i - y_i}{|x-y|^2}$.

Next, for $n \geq 3$ since we have

$$\begin{aligned} \frac{\partial}{\partial x_i}\left(\frac{1}{n(n-2)\omega_n} r^{2-n} \right) &= \frac{\partial r}{\partial x_i} \frac{d}{dr} \left(\frac{1}{n(n-2)\omega_n} |x-y|^{2-n} \right) \\ &= \frac{x_i - y_i}{r^2} \frac{1}{n(n-2)\omega_n} (2-n) r^{1-n}, \end{aligned}$$

it follows that

$$\frac{\partial \Gamma}{\partial \nu_x}(x,y) = \frac{-1}{n\omega_n} \frac{\rho^{1-n}\rho^2}{\rho^2}.$$

In fact, since $2\omega_2 = 2\pi$, we have for all $n \geq 2$ that

$$\frac{\partial \Gamma}{\partial \nu_x}(x,y) = \frac{-1}{n\omega_n \rho^{n-1}}.$$

Thus

$$\int_{\partial B_\rho(y)} \Gamma(x,y) \frac{\partial u}{\partial \nu}(x) dS(x) = \frac{-1}{n\omega_n \rho^{n-1}} \int_{\partial B_\rho(y)} u(x) dS(x) = -u(y),$$

by Theorem 2.4.1.

Hence combining all of this we see that as $\rho \to 0$ on boths sides of

$$\begin{aligned} \int_{\Omega \setminus B_\rho(y)} \Gamma(x,y) \Delta u(x) dx &= \int_{\partial \Omega} \left(\Gamma(x,y) \frac{\partial u}{\partial \nu}(x) - u(x) \frac{\partial \Gamma}{\partial \nu_x}(x,y) \right) dS(x) \\ &\quad + \int_{\partial B_\rho(y)} \left(\Gamma(x,y) \frac{\partial u}{\partial \nu}(x) - u(x) \frac{\partial \Gamma}{\partial \nu_x}(x,y) \right) dS(x), \end{aligned}$$

we obtain

$$\int_\Omega \Gamma(x,y) \Delta u(x) dx = \int_{\partial \Omega} \left(\Gamma(x,y) \frac{\partial u}{\partial \nu}(x) - u(x) \frac{\partial \Gamma}{\partial \nu_x}(x,y) \right) dS(x) + u(y),$$

which is what we wanted. ▶

2.19 Definition

Let Ω be a C^1 domain. A function $G(x,y)$ defined for $x, y \in \bar{\Omega}, x \neq y$ is call a **Green's function for** Ω if $G(x,y) = 0$ for $x \in \partial\Omega$ and $h(x,y) := G(x,y) - \Gamma(x,y)$ is harmonic for $x \in \Omega$ (and thus in particular at the point $x = y$). Since $h(x,y)$ is harmonic with prescribed boundary values, Corollary 2.8 forces $h(x,y)$, and thus also $G(x,y)$ to be unique (if it exists).

2.20 Green's functions and the representation formula

If a Greens' function $G(x,y)$ for Ω exists (which it does for Ω a C^1 domain, although we do not prove this) then putting $w(x) = h(x,y)$ in Green's formula (3) we obtain

$$-\int_\Omega (G(x,y) - \Gamma(x,y))\,\Delta u(x)dx = \int_{\partial\Omega} \left(u(x)\frac{\partial G}{\partial \nu_x}(x,y) - u(x)\frac{\partial \Gamma}{\partial \nu_x}(x,y) + \Gamma(x,y)\frac{\partial u}{\partial \nu}(x) \right) dS(x),$$

and so combining this with Theorem 2.18 we obtain

$$u(y) = \int_{\partial\Omega} u(x)\frac{\partial G}{\partial \nu_x}(x,y)dS(x) + \int_\Omega G(x,y)\Delta u(x)dx.$$

Note that this shows in particular this proves:

2.21 Corollary

If u is harmonic on a domain Ω which admits a Greens' function then u is determined by its values on $\partial\Omega$.

2.22 The Green's function of a ball

We wish to the compute the Green's function of the ball $B_\rho(0)$.

Given $y \in \mathbb{R}^n$, define

$$y^* = \begin{cases} \frac{\rho^2 y}{|y|^2} & y \neq 0 \\ \infty & y = 0. \end{cases}$$

Thus $y*$ is the point obtained by inversion in $\partial B_\rho(0)$. Now define

$$G(x,y) = \begin{cases} \Gamma(|x-y|) - \Gamma\left(\frac{|y|}{\rho}|x-y^*|\right) & y \neq 0 \\ \Gamma(|x|) - \Gamma(\rho) & y = 0. \end{cases}$$

Then $h(x,y) = \Gamma\left(\frac{|y|}{\rho}|x-y^*|\right)$ is harmonic (with respect to x) in $B_\rho(0)$, since if $y \in B_\rho(0)$ then $y^* \notin \overline{B_\rho(0)}$. As $y \to 0$ we have

$$\frac{|y|}{\rho}|x-y^*| = \frac{|y|}{\rho}|\left|x - \frac{\rho^2 y}{|y|^2}\right| \to \rho$$

and thus $G(x,y)$ is continuous. Finally, the formula

$$G(x,y) = \Gamma\left(\left(|x|^2 + |y|^2 - 2x\cdot y\right)^{\frac{1}{2}}\right) - \Gamma\left(\left(\frac{|x|^2|y|^2}{\rho^2} + \rho^2 - 2x\cdot y\right)^{\frac{1}{2}}\right)$$

shows that for $x \in \partial B_\rho(0)$ (so $|x| = \rho$) we have $G(x,y) = 0$. Thus $G(x,y)$ is indeed the Green's function for $B_\rho(0)$.

We conclude this chapter by quoting the following useful result.

2.23 Theorem (Poisson's integral formula)

Let $g \in C^0(\partial B_\rho(0))$. Then the function u defined by

$$u(y) := \begin{cases} \frac{\rho^2 - |y|^2}{n\omega_n \rho} \int_{\partial B_\rho(0)} \frac{g(x)}{|x-y|^n} dS(x) & y \in B_\rho(0) \\ g(y) & y \in \partial B_\rho(0), \end{cases}$$

is harmonic (and hence smooth, by Weyl's Lemma (Theorem 2.12) in $B_\rho(0)$ and continuous on $\overline{B_\rho(0)}$.

This allows us to explicitly solve the Dirichlet problem for the Laplacian on a ball; we shall use this a lot in the sequel.

3 Perron's method

3.1 Definition

Let Ω be a domain and $v \in C^0(\Omega)$. We say that v is **subharmonic** if for every subdomain $\mathcal{O} \subset\subset \Omega$ and for every harmonic function $u : \mathcal{O} \to \mathbb{R}$ that is in addition continuous on $\mathcal{O}$ such that $v \leq u$ on $\partial\mathcal{O}$ we also have $v \leq u$ on $\mathcal{O}$. We will often call the 'old' definition **classically subharmonic** from now on, that is, $u \in C^2(\Omega)$ is classically subharmonic if $\Delta u \geq 0$. We also define **superharmonic** continuous functions by reversing all the inequalities.

3.2 Proposition

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. A function $v \in C^0(\Omega)$ is subharmonic if and only if for every ball $B_\rho(y) \subseteq \Omega$ we have

$$v(y) \leq \frac{1}{n\omega_n\rho^{n-1}} \int_{\partial B_\rho(y)} v(x)dS(x),$$

or equivalently

$$v(y) \leq \frac{1}{n\omega_n\rho^n} \int_{B_\rho(y)} v(x)dx.$$

◀ If v is subharmonic, then given $B_\rho(y) \subseteq \Omega$, by the Poisson integral formula (Theorem 2.23) there exists a harmonic function $u : B_\rho(y) \to \mathbb{R}$ such that $u = v$ on $\partial B_\rho(y)$. Then since $v = u$ on $\partial B_\rho(y)$ we have

$$\frac{1}{n\omega_n\rho^{n-1}} \int_{\partial B_\rho(y)} v(x)dS(x) = \frac{1}{n\omega_n\rho^{n-1}} \int_{\partial B_\rho(y)} u(x)dS(x).$$

Moreover since v is subharmonic we have

$$v(y) \leq u(y) = \frac{1}{n\omega_n\rho^{n-1}} \int_{\partial B_\rho(y)} u(x)dS(x) = \frac{1}{n\omega_n\rho^{n-1}} \int_{\partial B_\rho(y)} v(x)dS(x).$$

For the converse, we note that the proof of the strong maximum principle for harmonic functions (Corollary 2.6) used no properties of harmonic functions apart from the mean-value properties (Theorem 2.4). Hence if v satisfies the hypotheses, and u is harmonic such that $v \leq u$ on $\partial B_\rho(y)$ then $v - u$ satisfies the strong maximum principle, and hence $v \leq u$ on $B_\rho(y)$. ▶

3.3 Lemma (Properties of subharmonic functions)

1. For $u \in C^2(\Omega)$, we have u subharmonic if and only if u is classically subharmonic, that is, $\Delta u \geq 0$ on Ω. Thus the new definition is a genuine enlargement of the class of (classically) subharmonic functions.

2. **(Strong maximum prinicple for subharmonic functions)** If v is subharmonic in Ω and there exists $x_0 \in \Omega$ such that $v(x_0) = \sup_\Omega v(x)$ then v is constant. In particular, if $v \in C^0(\bar{\Omega})$ then $v(x) \leq \max_{\partial\Omega} v(y)$ for all $x \in \Omega$.

3. **(Harmonic replacements)** If $v \in C^0\left(\bar{\Omega}\right)$ is subharmonic and $B_\rho(y) \subset\subset \Omega$ then the **harmonic replacement** V of v, defined by

$$V(x) := \begin{cases} v(x) & x \in \Omega \backslash B_\rho(y), \\ \frac{\rho^2 - |x-y|^2}{n\omega_n\rho} \int_{\partial B_\rho(y)} \frac{v(x)}{|z-x|^n} dS(z) & x \in B_\rho(y), \end{cases}$$

   is subharmonic in Ω and harmonic in $B_\rho(y)$.

4. If $v_1, \ldots, v_n$ are subharmonic then so is v defined by $v(x) := \max_i\{v_i(x)\}$.

◀ To prove 1, first suppose $u \in C^2(\Omega)$ is classically subharmonic. Suppose $B_\rho(y) \subseteq \Omega$. Choose $0 < r < \rho$. Then from the proof of Theorem 2.4.1 we have

$$0 \leq \frac{1}{n\omega_n r^{n-1}} \int_{B_r(y)} \Delta u(x) dx = \frac{\partial}{\partial r}\left(\frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(y)} u(x) dS(x) \right) = \phi'(r),$$

and integrating this gives $\phi(r) \leq \phi(\rho)$. Letting $r \to 0$, we have $\phi(r) \to u(y)$ and thus

$$u(y) \leq \phi(\rho) = \frac{1}{n\omega_n \rho^{n-1}} \int_{\partial B_\rho(y)} u(x) dS(x),$$

which verifies the hypotheses of Proposition 3.2. Hence u is subharmonic. For the converse, suppose $u \in C^2(\Omega)$ is subharmonic but $\Delta u(y) < 0$ for some $y \in \Omega$. Then as $u \in C^2(\Omega)$, we can find a ball $B_\rho(y) \subseteq \Omega$ such that $\Delta u(x) < 0$ for $x \in B_\rho(y)$. Applying the first part of the proof to $-u$, we conclude

$$u(y) > \frac{1}{n\omega_n \rho^{n-1}} \int_{\partial B_\rho(y)} u(x) dS(x),$$

which contradicts the strong maximum principle (Corollary 2.6).

Next, 2 is immmediate from Proposition 3.2. To prove 3, first observe that $v \leq V$ as v is subharmonic. Let $\mathcal{O} \subset\subset \Omega$, and u harmonic on $\mathcal{O}$ and continuous on $\bar{\mathcal{O}}$, with $V \leq u$ on $\partial\mathcal{O}$. Then also $v \leq u$ on $\partial\mathcal{O}$, and hence $v \leq u$ on $\mathcal{O}$. Thus $V \leq u$ on $\mathcal{O}\backslash B_\rho(y)$. Hence $V \leq u$ on $\mathcal{O} \cap \partial B_\rho(y)$. Since V is harmonic (and thus in particular, subharmonic) on $\mathcal{O} \cap B_\rho(y)$, we have $V \leq u$ on $\mathcal{O} \cap B_\rho(y)$. Hence $V \leq u$ on $\mathcal{O}$, which shows V is subharmonic.

Finally, to prove 4, let $\mathcal{O} \subset\subset \Omega$, u harmonic on $\mathcal{O}$ and continuous on $\bar{\mathcal{O}}$, and $v \leq u$ on $\partial\mathcal{O}$. Then for all i, $v_i \leq u$ on $\partial\mathcal{O}$, and hence $v_i \leq u$ on $\mathcal{O}$, and thus the same is true of v. ▶

The next result is the main one of this chapter, and it will take the rest of the chapter in order to prove it.

3.4 Theorem (Perron's Method)

Let $\Omega \subseteq R^n$ be a C^2 domain (the result still holds if we do not assume such strong boundary regularity on Ω, but the proof is harder). Let $g \in C^0(\partial\Omega)$. Then the **Dirichlet problem for the Laplacian** is solvable in Ω, that is we can solve

$$\Delta u = 0 \text{ in } \Omega, u = g \text{ on } \partial\Omega.$$

The proof of Theorem 3.4 is in two stages. The first is based on the following definition.

3.5 Definition

A subharmonic function $v \in C^0\left(\bar{\Omega}\right)$ is called a **subfunction** with respect to g if $v \leq g$ on $\partial\Omega$. Let S_g be the set of all subfunctions with respect to g. Thus S_g is the set of all **subsolutions** to the problem in hand.

3.6 Proposition

Define

$$u(x) := \sup_{v \in S_g} v(x).$$

Then u is harmonic on Ω.

◀ First we check u is well defined. We must show that S_g is non-empty, and that elements of S_g are uniformly bounded. Choose $0 < c < \inf_{\partial\Omega} g \leq \sup_{\partial\Omega} g < C$. Then the constant function c lies in S_g, and by the strong maximum principle (Lemma 3.3.2), if $v \in S_g$ then $v \leq C$.

Now let $y \in \Omega$ be arbitrary. By definition of $u(y)$, there exists a sequence $\{v_n\}$ of members of S_g such that $v_n(y) \to u(y)$ from below. Moreover, replacing v_n by $\max\{v_0, v_n\}$ (valid by Lemma

3.3.4) we may assume without loss of generality that the $\{v_n\}$ are bounded below. Pick $R > 0$ such that $\overline{B_{2R}(y)} \subseteq \Omega$. Now consider the sequence $\{V_n\}$, where V_n is the harmonic replacement of v_n in $B_{2R}(y)$. Note that the (V_n) are bounded uniformly above and below in $B_R(y)$ by the maximum (and the corresponding minimum - harmonic functions are both subharmonic and superharmonic) principle. Now by Proposition 2.13.2 we have gradient bounds on the V_n; specifically we have $\sup_{n\in\mathbb{N}} |DV_n| \leq C$. Then by the mean-value theorem, for any n and $x, z \in B_R(y)$ we have $|V_n(x) - V_m(z)| \leq C|x - z|$ and hence the sequence $\{V_n\}$ is equicontinuous. Thus by the Arzela-Ascoli theorem, passing to a subseqence if neccesary, we may assume that the V_n converge uniformly to a function V on $B_R(y)$. By Corollary 2.16, V is harmonic. The claim now is that $V = u$.

First, since $V_n \in S_g$ by Lemma 3.3.1, we have $v_n \leq V_n \leq u$ by the maximum priniciple (since $V_n = v_n$ on $\partial B_R(y)$ and hence $V_n \leq v_n$ on $B_R(y)$ definition of u. But then

$$V(y) = \lim_{n\to\infty} V_n(y) \geq \lim_{n\to\infty} v_n(y) = u(y) \geq V(y).$$

Note also by definition we have $V \leq u$ on $B_R(y)$ as the same is true of all the (V_n). We thus have $V \leq u$ on $B_R(y)$ and $V(y) = u(y)$. We will now show that $V = u$ on $B_R(y)$. Indeed, suppose $V(z) < u(z)$ for some $z \in B_R(y)$. Then by assumption we can find $w \in S_g$ such that $V(z) < w(z) \leq u(z)$. Now define $w_n = \max\{w, V_n\}$, and then let W_n be the harmonic replacement of w_n in $B_{2R}(y)$. By the reasoning above, passing to a subsequence if necessary we may assume that the (W_n) converge uniformly to some harmonic function W on $B_R(y)$.

Then since $v_n \leq w_n$ and $w_n \in S_g$, the maximum principle implies $V \leq W \leq u$ on $B_R(y)$. But since $W(y) \leq u(y) = V(y)$, by applying the strong maximum prinicple for harmonic functions (Corollary 2.6) to $W - V$ we conclude that $W - V$ has an interior maximum and hence is a constant, which is $W(y) - V(y) = 0$. The contradiction is then obtained, as

$$W(z) = \lim_{n\to\infty} W_n(z) = \lim_{n\to\infty} (\max\{w(z), V_n(z)) \geq w(z) > V(z) = W(z).$$

This completes the proof of the proposition. ▶

The second stage of the proof of Perron's method (Theorem 3.4) requires the following definition.

3.7 Definition

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain. Let $\xi \in \partial\Omega$. A function $\beta \in C^0\left(\bar{\Omega}\right)$ is called a **barrier** at ξ with respect to Ω if:

1. $\beta > 0$ in $\bar{\Omega}\backslash\{\xi\}$ and $\beta(\xi) = 0$.
2. β is superharmonic in Ω.

$\xi \in \partial\Omega$ is called **regular** if there exists a barrier β at ξ with respect to $\partial\Omega$.

3.8 Proposition

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. The Dirichlet problem

$$\Delta u = 0 \text{ in } \Omega, u = g \text{ on } \partial\Omega,$$

is solvable for all continuous $g \in C^0(\partial\Omega)$ if and only if every point $\xi \in \partial\Omega$ is regular.

First the easy direction. If it is solvable for all continuous g, given $\xi \in \partial\Omega$, let $g(x) = |x - \xi|$. Then the solution u for that g is a barrier at ξ with respect to Ω, since $u(\xi) = g(\xi) = 0$, and since $\min_{\partial\Omega} g(x) = 0$, by the stong maximum prinicple, $u > 0$. Hence ξ is regular.

For the converse, we will show that if $\xi \in \partial\Omega$ is regular then

$$\lim_{x\in\Omega\to\xi} u(x) = g(\xi).$$

Since by Proposition 3.2 we know u to be harmonic in Ω, it follows that u is then the desired solution.

So let $\xi \in \partial\Omega$ be regular and β a barrier at ξ with respect to Ω. Let $M := \max_{\partial\Omega} |g(x)|$. Continuity of g at ξ implies for every $\epsilon > 0$ there exists $\delta > 0$ and a constant $C = C(\epsilon)$ such that

$$|g(x) - g(\xi)| < \epsilon \text{ for } |x - \xi| < \delta,$$

$$C\beta(x) \geq 2M \text{ for } |x - \xi| \geq \delta.$$

Now define

$$s^+(x) = g(\xi) + \epsilon + C\beta(x),$$
$$s^-(x) = g(\xi) - \epsilon - C\beta(x).$$

Then s^+ is superharmonic and s^- is subharmonic in Ω. By choice of C, δ we have $s^- \leq g \leq s^+$ in $\partial\Omega$. Thus s^- is a subsolution and s^+ is a supersolution. Thus $s^- \leq u$ in Ω as $s^- \in S_g$, and moreover if $v \in S_g$ then by the maximum principle we have $v \leq s^+$ in Ω. Taking the supremum over such v, we obtain $u \leq s^+$ in Ω. Hence for all $x \in \Omega$, we have

$$|u(x) - g(\xi)| \leq \epsilon + C\beta(x).$$

Since $\lim_{x\to\xi} \beta(x) = 0$, the result follows. ▶

We now give a sufficient condition for every point of the boundary of a domain Ω to be regular.

3.9 Definition

A domain $\Omega \subseteq \mathbb{R}^n$ satisfies an **exterior sphere condition** at $\xi \in \partial\Omega$ if there exists $y \in \mathbb{R}^n$ and $\rho > 0$ such that $\overline{B_\rho(y)} \cap \overline{\Omega} = \{\xi\}$.

3.10 Lemma

If Ω satisfies an exterior sphere condition at ξ, then $\partial\Omega$ is regular at ξ.

◀ Define

$$\beta(x) := \begin{cases} \rho^{2-n} - |x-y|^{2-n} & n \geq 3 \\ \log \frac{|x-y|}{\rho} & n = 2. \end{cases}$$

Then $\beta(\xi) = 0$, and β is harmonic in $\mathbb{R}^n \backslash \{y\}$, and hence harmonic in Ω. Since for $x \in \overline{\Omega}\backslash\{\xi\}$, we have $|x - y| > \rho$, and hence $\beta(x) > 0$ for all $x \in \overline{\Omega}\backslash\{\xi\}$. ▶

Finally, to complete the proof of Perron's method we show that C^2 domains satisfy an exterior sphere condition at every point.

3.11 Lemma

If $\Omega \subseteq \mathbb{R}^n$ is a C^2 domain then Ω satisfies an exterior sphere condition at every point $\xi \in \partial\Omega$.

◀ Let $\xi \in \partial\Omega$. Then there exists $R > 0$ and a C^2 function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ such that after relabelling and reorientating the coordinates if necessary, $\Omega \cap B_R(\xi) = \{x \in B_R(\xi) | x_n < f(x_1, \ldots, x_n)\}$ (and so $\partial\Omega \cap B_R(\xi) = \{x \in B_R(\xi) \mid x_n = f(x_1, \ldots, x_n)\}$). Furthermore we may assume that ξ is the origin in this coordinate system (so $f(0) = 0$), and the tangent plane to $\partial\Omega$ at ξ is the plane $\{x_n = 0\}$. If ν is the outward unit normal to Ω at ξ, so ν_n is positive, and in our coordinate system $\nu = (0, 0, \ldots, 0, 1)$. Now set $y = \xi + \delta\nu$, so in our coordinate system y is the point $(0, 0, \ldots, 0, \delta)$. We claim that for suitably small δ, the ball $\overline{B_\delta(y)}$ intersects $\overline{\Omega}$ only at ξ. More precisely, we will prove that $x \in B_\delta(y)$ only if $x \in B_R(\xi)$ and $x_n > f(x_1, \ldots, x_n)$, which suffices.

By Taylor's theorem,

$$f(x_1,\ldots,x_{n-1}) = f(0) + \sum_{i=1}^{n-1} D_i f(0) x_i + O(x_1^2+\cdots+x_{n-1}^2) = 0 + O(x_1^2,\ldots,x_{n-1}^2),$$

as all the first-order deriviatives must vanish at the origin since the tangent plane to $\partial\Omega$ at the origin is the plane $\{x_n = 0\}$. Hence there exists a constant $M \geq 0$ such that

$$|f(x_1,\ldots,x_{n-1})| \leq M(x_1^2+\cdots+x_{n-1}^2)$$

for $x \in B_R(\xi)$. Now if $x \in B_\delta(y)$ then

$$|x-y|^2 = x_1^2+\cdots+x_n^2 - 2\delta x_n + \delta^2 < \delta^2,$$

that is $x \in B_\delta(y)$ if and only if $|x|^2 < 2\delta x_n$. Now choose $\delta \leq \min\{\frac{M}{2}, \frac{\epsilon}{2}\}$. Then if $x \in B_\delta(y)$ we have

$$f(x_1,\ldots,x_{n-1}) \leq M(x_1^2+\cdots+x_{n-1}^2) \leq M|x|^2 < 2M\delta x_n \leq x_n,$$

which completes the proof. ▶

With this, Perron's method (Theorem 3.4) is of course proved.

4 General second order linear elliptic operators

4.1 Definition

We now proceed to study general **second order linear** operators L of the form

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u(x) + \sum_{j=1}^{n} b_j(x) D_j u(x) + c(x) u(x), \tag{6}$$

for $x \in \Omega \subseteq \mathbb{R}^n$, say. In general we will omit the summation sign and write

$$L = a_{ij} D_{ij} + b_j D_j + c.$$

We will generally assume u is C^2, and hence by setting $a'_{ij}(x) = \frac{a_{ij}(x)+a_{ji}(x)}{2}$ we may assume the matrix valued function $A(x) = (a_{ij}(x))$ is symmetric (this works as $D_{ij}u = D_{ji}u$).

We say L is **elliptic** if $A(x)$ is positive definite for all $x \in \Omega$. By **Rayleigh's quotient** this is equivalent to the smallest eigenvalues $\lambda_{\min}(x)$ satisfying

$$\lambda_{\min}(x) = \min_{|\xi|\neq 0} \frac{a_{ij}(x)\xi_i\xi_j}{|\xi|^2} > 0$$

(here we are summing over i and j).

We say that L is **uniformly elliptic** if there exists a positive $\lambda > 0$ such that for all $x \in \Omega$ and all $\xi \in \mathbb{R}^n$,

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2.$$

We call λ the **constant of uniform ellipticity**.

4.2 Example

We cannot hope for a maximum principle or a minimum principle for every elliptic operator of the form in (6).

Indeed, if $\Omega = (0,\pi) \subseteq \mathbb{R}$ then

$$u(x) = \sin x$$

satisfies the Dirichlet problem

$$\Delta u + u = 0 \text{ in } \Omega, u = 0 \text{ on } \partial\Omega.$$

However u has an interior maximum at $\pi/2$, and if $u(x) = -\sin x$ then it has an interior minimum at $\pi/2$.

Similarly, taking $\Omega = (0,\pi) \times (0,\pi) \subseteq \mathbb{R}^2$, $u(x,y) = \sin x \sin y$ solves the Dirichlet problem for the Laplacian with zero boundary values, but has an interior maximum.

However, if $\lambda \leq 0$, the Dirichlet problem

$$\Delta u + \lambda u = 0 \text{ in } \Omega, u = 0 \text{ on } \partial\Omega$$

has no non-zero solutions, as multiplying by u yields $u\Delta u + \lambda u^2 = 0$, and since u vanishes on the boundary, when we integrate by parts the boundary terms disappear to give

$$0 = \int_\Omega u\Delta u + \lambda u^2 = \int_\Omega |Du|^2 - \lambda \int_\Omega u^2,$$

which forces $u = 0$.

This is not surprising, since we have:

4.3 Theorem (Weak maximum principle for uniformly elliptic operators)

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, and $L = a_{ij}D_{ij} + b_jD_j + c$ be a uniformly elliptic operator with constant of uniform ellipticity $\lambda > 0$, and suppose that $a_{ij}(x)$, $b_j(x)$ and $c(x)$ are all bounded functions, and (without loss of generality) the matrix $A(x) = [a_{ij}(x)]$ is symmetric.

Suppose $u \in C^2(\Omega) \cap C^0\left(\bar{\Omega}\right)$ is a **subsolution**, that is, $Lu \geq 0$ in Ω.

Then if $c(x) \leq 0$ for all $x \in \Omega$ we have

$$\sup_\Omega u \leq \max_{\partial\Omega} u^+,$$

where $u^+(x) := \max\{u(x), 0\}$. If $c = 0$ then we have

$$\sup_\Omega u = \max_{\partial\Omega} u.$$

◄ First we prove the special case where $Lu > 0$ in Ω. Suppose $c(x) \leq 0$, and suppose x_0 is an interior non-negative maximum for u. Then $Du(x_0) = 0$ and the Hessian matrix $D^2u(x_0)$ is negative semi-definite. Since the matrix $A(x_0) = [a_{ij}(x_0)]$ is positive definite, it follows that the matrix product $A(x_0) \cdot D^2u(x_0)$ is negative semi-definite, and thus has a non-positive trace. But

$$\operatorname{trace}(A(x_0) \cdot D^2u(x_0)) = a_{ij}(x_0)D_{ij}(x_0) \geq Lu(x_0) > 0;$$

this is a contradiction. Thus u has no non-negative interior maxima, and thus

$$\sup_\Omega u \leq \max_{\partial\Omega} u^+.$$

If $c = 0$, then we do non need to assume x_0 is a non-negative maximum of u for the above proof to go through, and hence we obtain

$$\sup_\Omega u = \max_{\partial\Omega} u$$

For the general case, set $v(x) = u(x) + \epsilon e^{\gamma x_1}$ (where x_1 is the first coordinate of x), and $\gamma > 0$ is a constant that we will choose later.

Then $Lv(x) = Lu(x) + \epsilon L(e^{\gamma x_1}(x))$. Moreover,

$$Le^{\gamma x_1}(x) = e^{\gamma x_1}(a_{11}(x)\gamma^2 + \gamma b_1(x) + c(x)).$$

Now setting $\xi = (1, 0, \ldots, 0)$ we see that $a_{11}(x) \geq \lambda$, and hence $Le^{\gamma x_1}(x) \geq e^{\gamma x_1}(\lambda\gamma^2 + \gamma b_1(x) + c(x))$. Since b_1 and c are bounded, we may choose γ large enough such that $Le^{\gamma x_1}(x) > 0$ for all $x \in \Omega$ (note that γ only depends on $\|b_1\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}$ and λ). Thus if $c(x) \leq 0$, we obtain

$$\sup_\Omega v \leq \max_{\partial\Omega} v^+,$$

and hence
$$\sup_\Omega u \le \sup_\Omega v \le \max_{\partial\Omega} v^+ \le \max_{\partial\Omega} u^+ + \epsilon \max_{\partial\Omega} e^{\gamma x_1}.$$
But ϵ was arbitrary, and thus the result follows by letting $\epsilon \to 0$. If $c = 0$, then we similarly obtain
$$\sup_\Omega u \le \sup_\Omega v = \max_{\partial\Omega} v = \max_{\partial\Omega} u + \epsilon \max_{\partial\Omega} e^{\gamma x_1},$$
and again letting $\epsilon \to 0$ completes the proof. ►

4.4 Corollary

In the above situation, if instead $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a **supersolution**, that is, $Lu \le 0$ in Ω then if $c(x) \le 0$ for all $x \in \Omega$ we have
$$\inf_\Omega u \ge \min_{\partial\Omega} u^-,$$
where $u^-(x) := \min\{u(x), 0\}$. If $c = 0$ then we have
$$\inf_\Omega u = \min_{\partial\Omega} u.$$

◄ Apply Theorem 4.3 to $-u$. ►

4.5 Corollary

If $u \in C^2(\Omega) \cap C^0\left(\bar{\Omega}\right)$ is a solution, that is $Lu = 0$ in Ω then if $c(x) \le 0$ for all $x \in \Omega$ then
$$\sup_\Omega |u| = \max_{\partial\Omega} |u|.$$

◄ By Theorem 4.3 for any $x \in \Omega$, $u(x) \le \max_{\partial\Omega} u^+ \le \max_{\partial\Omega} |u|$, and also $-u(x) \le \max_{\partial\Omega} u^+ \le \max_{\partial\Omega} |u|$. Hence $|u(x)| \le \max_{\partial\Omega} |u|$. Taking the supremum over $x \in \Omega$ gives the result. ►

4.6 Example

These maximum principles are specific to second order equations. For example, taking $\Omega = (0, 1)$ and defining
$$Lu = \frac{\partial^4 u}{\partial x^4},$$
we find that if we insist on boundary conditions $u(0) = 0$,$u(1) = -1$ then $u(x) = 3x^2 - 4x^3$ has a stricitly positive maximum at $x = 1/2$.

The next result is another 'maximum principle' style result, this time taking advantage of the shape of the domain Ω.

4.7 Proposition

Assume $Lu = a_{ij} D_{ij} u + b_j D_j u + cu$ is unifomly elliptic, with uniform ellipticity constant $\lambda > 0$, and assume a_{ij}, b_j and c are bounded functions $\Omega \to \mathbb{R}$, where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain contained in a strip
$$\{x \in \mathbb{R}^n \mid |x_i| \le d\}.$$
Suppose $u \in C^2(\Omega) \cap C^0\left(\bar{\Omega}\right)$ is a subsolution to $Lu = f$, where $f : \Omega \to \mathbb{R}$ is a bounded function. Let
$$\beta := \sup_\Omega \frac{d|b_1| + d^2|c|}{\lambda},$$
and set
$$C := \frac{\left(e^{2(1+\beta)} - 1\right) d^2}{\lambda}.$$

Then if $c \le 0$ in Ω we have

$$\sup_{\Omega} u \le \max_{\partial\Omega} u^{+} + C \sup_{\Omega} |f|,$$

and if $c \equiv 0$ in Ω then we have

$$\sup_{\Omega} |u| \le \max_{\partial\Omega} |u| + C \sup_{\Omega} |f|.$$

◀ To simplify the problem, we rescale. Set $\tilde{a}_{ij}(x) = \lambda^{-1} a_{ij}(dx)$,$\tilde{b}_j(x) = \lambda^{-1} d b_j(dx)$, $\tilde{c}(x) = \lambda^{-1} d^2 c(dx)$ and $\tilde{f}(x) = \lambda^{-1} d^2 f(x)$. Set $\tilde{\Omega} = \{x \in \mathbb{R}^n \mid dx \in \Omega\}$, and define $\tilde{u}(x) = u(dx)$. Then if $\tilde{L}(\cdot) = \tilde{a}_{ij} D_{ij}(\cdot) + \tilde{b}_j(\cdot) + \tilde{c}(\cdot)$ we see that $Lu \ge f$ in Ω if and only if $\tilde{L}\tilde{u} \ge \tilde{f}$ in $\tilde{\Omega}$. Moreover, $\tilde{\Omega} \subseteq \{x \in \mathbb{R}^n \mid |x_1| \le 1\}$ and the uniform ellipticity constant of $(\tilde{a}_{ij})$ is 1. In other words, without loss of generality we may assume that $d = \lambda = 1$.

Thus we have $\beta = \sup_{\Omega} \{|b_1(x)| + |c(x)|\}$ and $C = e^{2(1+\beta)} - 1$. Now if $M = \sup_{\Omega} |f|$, set

$$w(x) = M\left(e^{\gamma(1+x_1)} - 1\right),$$

where $\gamma > 0$ is some positive constant to be chosen later.

Observe

$$Lw = \left(\sup_{\Omega} |f|\right) L\left(e^{\gamma(1+x_1)} - 1\right) = M\left(e^{\gamma(1+x_1)}\left(\gamma^2 a_{11}(x) + \gamma b_1(x) + c(x)\right) - c(x)\right),$$

which, for $x \in \Omega$ (so $|x_1| \le 1$) gives

$$Lw \ge M\left(\gamma^2 - |b_1|\gamma\right),$$

where we have used the fact that $\mu = 1$, and that $|a_{11}| \le \mu$ (cf. (9.4)). Now if we choose $\gamma = 1 + \beta$ we have $Lw \ge M$.

Now set $v(x) = u(x) + w(x)$. Then $Lv \ge 0$, and hence by the weak maximum principle (Theorem 4.3) if $c \le 0$ we have

$$\sup_{\Omega} v \le \max_{\partial\Omega} v^{+},$$

and hence

$$\sup_{\Omega} u \le \sup_{\Omega} v \le \max_{\partial\Omega} v^{+} = \max_{\partial\Omega} u^{+} + \max_{\partial\Omega} w^{+} = \max_{\partial\Omega} u^{+} + M\left(e^{2(1+\beta)} - 1\right),$$

as required. Finally if $c \equiv 0$, then Theorem 4.3 gives us

$$\sup_{\Omega} |v| = \max_{\partial\Omega} |v|,$$

and hence

$$\sup_{\Omega} |u| \le \sup_{\Omega} |v| = \max_{\partial\Omega} |v| \le \max_{\partial\Omega} |u| + \max_{\partial\Omega} |w| = \max_{\partial\Omega} |u| + M\left(e^{2(1+\beta)} - 1\right).$$

This completes the proof. ▶

We now wish to prove a form of the strong maximum principle for operators of the form (6).

4.8 Theorem (Hopf's strong maximum principle)

Let $\Omega \subseteq \mathbb{R}^n$ be a domain (not necessarily bounded), and $Lu = a_{ij} D_{ij} u + b_j u + cu$ a uniformly elliptic operator in Ω with a_{ij}, b_j and c bounded functions. Suppose $Lu \ge 0$ in Ω, with $u \in C^2(\Omega) \cap C^0\left(\bar{\Omega}\right)$.

Then if $c \le 0$, u cannot attain a non-negative maximum in Ω unless it is constant. If $c \equiv 0$, u cannot attain a maximum in Ω unless it is constant. If x_0 is an interior maximum or an interior minimum such that $u(x_0) = 0$ then $u \equiv 0$ irrespective of the sign of c.

In order to prove this we need the following important result.

4.9 Proposition (Hopf's boundary point lemma)

Let $u \in C^2(B_R(y)) \cap C^0\left(\overline{B_R(y)}\right)$ and L as in the statement of Theorem 4.8. Suppose $Lu \geq 0$, and suppose there exists $x_0 \in \partial B_R(y)$ such that $u(x) < u(x_0)$ for all $x \in \overline{B_R(y)} \backslash \{x_0\}$. Then if any of the three following conditions hold:

1. $c \leq 0$ in $B_R(y)$ and $u(x_0) \geq 0$,
2. $c \equiv 0$ in $B_R(y)$,
3. $u(x_0) = 0$,

then

$$\liminf_{x \to x_0} \frac{u(x_0) - u(x)}{|x - x_0|} > 0,$$

where the angle between the vector $x - x_0$ and the normal at x_0 is less than $\frac{\pi}{2} - \delta$ for some fixed $\delta > 0$.

In particular if $u \in C^2\left(B_R(y)\right) \cap C^1\left(\overline{B_R(y)}\right)$ then if $r(x) = |x - y|$ is an exterior normal to $\partial B_R(y)$ then

$$\frac{\partial u}{\partial r}(x_0) > 0.$$

◀ For $0 < \rho < R$, on the annular region $B_R(y) \backslash \overline{B_\rho(y)}$ we consider $v(x) := e^{-\gamma|x-y|^2} - e^{-\gamma R^2}$, we have

$$D_i v(x) = -2\gamma(x_i - y_i)e^{-\gamma|x-y|^2},$$

and

$$D_{ij} v(x) = (4\gamma^2(x_i - y_i)(x_j - y_j) - 2\gamma\delta_{ij})e^{-|x-y|^2},$$

and thus

$$\begin{aligned} Lv(x) \quad &= \quad \left(4\gamma^2 \sum_{i,j=1}^{n} a_{ij}(x)(x_i - y_i)(x_j - y_j) - 2\gamma \sum\left(_{i=1}^{n} a_{ii}(x) + b_i(x)(x_i - y_i)\right)\right) e^{-\gamma|x-y|^2} \\ &\quad +c(x)\left(e^{-\gamma|x-y|^2} - e^{-\gamma R^2}\right). \end{aligned}$$

For sufficiently large γ, because of the assumed boundedness of the coefficients of L and the ellipiticity condition, we have $Lv > 0$ in the annulus $B_R(y) \backslash \overline{B_\rho(y)}$. By assumption $u(x) - u(x_0) < 0$ in $\overline{B_R(y)}$ and thus by compactness there exists $\epsilon > 0$ such that $w(x) = u(x) - u(x_0) + \epsilon v(x) < 0$ for $x \in \partial B_\rho(y)$. Since $v = 0$ on $\partial B_R(y)$, this holds on $\partial B_R(y)$, too.

On the other hand,

$$Lw(x) \geq -Lu(x_0)(x) = -c(x)u(x_0) \geq 0$$

(this holds if either of the three possible hypotheses hold).

Thus by the weak maximum principle, it follows that $w(x) \leq 0$ for all $x \in B_R(y) \backslash \overline{B_\rho(y)}$. Provided the deriviative exists, we therfore have

$$\frac{\partial w}{\partial r}(x_0) \geq 0,$$

and hence,

$$\frac{\partial u}{\partial r}(x_0) \geq -\epsilon \frac{\partial v}{\partial r}(x_0) = \epsilon\left(2\gamma R e^{-\gamma R^2}\right) > 0.$$

In any case

$$\liminf_{x \to x_0} \frac{u(x_0) - u(x)}{|x - x_0|} > 0,$$

with the requirement stated on the angle between $x - x_0$ and the normal to x_0 clearly being satisfied. ▶

We now prove Theorem 4.8.

◀ **(Proof fo Theorem 4.8)** Assume u has a maximum M in Ω (if $c \neq 0$ and $M \neq 0$, we must in addition assume $M \geq 0$). Assume for contradiction that u is not constant. Then $\Sigma := \{x \in \Omega \mid u(x) < M\}$ is open and nonempty, and moreover $\partial\Sigma \cap \Omega \neq \emptyset$. Choose $y \in \Sigma$ such that y is strictly closer to $\partial\Sigma$ than $\partial\Omega$ (if Ω is not bounded this restriction is not necessary), and choose R maximal such that $B_R(y) \subseteq \Sigma$. If $\#\left\{\overline{B_R(y)} \cap \partial\Sigma\right\} > 1$, by slightly moving y towards one of the points of intersection, and shrinking R suitably, we may assume $\overline{B_R(y)} \cap \partial\Sigma = \{x_0\}$, say, Then we have $u(x_0) = M$ for $x_0 \in \partial B_R(y)$, and $u(x) < u(x_0)$ for $x \in B_R(y)$. By Hopf's boundary point lemma (Proposition 4.9), $Du(x_0) \neq 0$, which is a contradiction as x_0 is assumed to be a local interior maximum. ▶

5 Sobolev spaces

5.1 Motivation for Sobolev spaces

Let us return to to the Dirichlet problem for the Laplacian: let $\Omega \subseteq \mathbb{R}^n$ be a C^1 domain, and $g \in C^2\left(\bar{\Omega}\right)$. We wish to solve $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$. By Section 2.1, finding a C^2 solution u is equivalent to finding a minimizer of $\mathcal{F}(\cdot) = \int_\Omega |D\cdot|^2$ over the class

$$\mathcal{C} := \{v \in C^2(\Omega) \mid v = g \text{ on } \partial\Omega\}$$

(note $\mathcal{C} \neq \emptyset$, as $g \in \mathcal{C}$). In other words, we need $u \in \mathcal{C}$ such that

$$\mathcal{F}(u) = \inf_{v \in \mathcal{C}} \mathcal{F}(u).$$

We know by definition of the infimum that there exists a **minimising sequence** $\{v_j\} \in \mathcal{C}$ such that

$$\mathcal{F}(v_j) \to \inf_{v \in \mathcal{C}} \mathcal{F}(v).$$

What we want is some $u \in \mathcal{C}$ such that there exists a subsequence $\{v_{j'}\}$ with $v_{j'} \to u$ in some sense, such that under this convergence the functional $\mathcal{F}$ is **lower semi-continuous**, that is,

$$\mathcal{F}(\lim_{j' \to \infty} v_{j'}) \leq \lim_{j' \to \infty} \mathcal{F}(v_{j'})$$

(since the right-hand limit is assumed to be the infimum, we don't need continuity - lower semi-continuity is enough).

Unfortunately this will not necessarily hold in $\mathcal{C}$; we need a larger space.

5.2 The completion $\mathcal{S}$

Define an inner product on $\mathcal{C}$ by

$$\langle u, v\rangle := \int_\Omega Du \cdot Dv.$$

Under the associated norm $\|\cdot\|$, $\{v_j\}$ is Cauchy, as

$$\|v_j\|^2 = \int_\Omega |Dv_j|^2 = \mathcal{F}(v_j),$$

and thus since $\{\mathcal{F}\left(v_j\right)\}$ is convergent, it is in particular Cauchy. The natural thing to do therfore is to let $\mathcal{S}$ be the completion of $\mathcal{C}$ with respect to this norm ($\mathcal{C}$ is not already complete under this norm). Note $\mathcal{S}$ is then a Hilbert space.

5.3 Solving the Dirichlet problem for Poisson's equation

Now in the notation above set $f = -\Delta g$ and $w = u - g$. Then w solves $\Delta w = f$ in Ω, $w = 0$ on $\partial\Omega$ if and only if u solves $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$. Thus let

$$\mathcal{C}_0 := \left\{v \in C^2(\Omega) \mid v = 0 \text{ on } \partial\Omega\right\}$$

with completion $\mathcal{S}_0$. Consider the bounded linear functional $\mathcal{L}$ on $\mathcal{C}_0$ defined by

$$\mathcal{L}(\varphi) = -\int_\Omega f\varphi.$$

Observe that if w solves $\Delta w = f$ in Ω, $w = 0$ on $\partial\Omega$, then for any $\varphi \in \mathcal{C}_0$,

$$\int_\Omega Dw \cdot D\varphi = -\int_\Omega \Delta w\varphi = -\int_\Omega f\varphi = \mathcal{L}(\varphi),$$

as the boundary terms vanish since $\varphi, w = 0$ on $\partial\Omega$. In other words,

$$\langle w, \varphi\rangle = \mathcal{L}(\varphi).$$

Now by the Hahn-Banach theorem, $\mathcal{L}$ extends to a bounded linear functional $\tilde{\mathcal{L}}$ on $\mathcal{S}$, and since $\mathcal{S}$ is a Hilbert space, by the Riesz representation theorem, there exists some $\tilde{w} \in \mathcal{S}$ such that

$$\tilde{\mathcal{L}}(\varphi) = \langle \tilde{w}, \varphi\rangle$$

for all $\varphi \in \mathcal{S}$. From what we have shown, this $\tilde{w}$ is a **generalised solution** to the problem. Thus by working in $\mathcal{S}$ we can indeed find a possible solution. The challenge is then to try and show that actually $\tilde{w} \in \mathcal{C}$, and hence is the actual solution we are looking for. This is a tricky problem, and depends on f and Ω.

The problem with this approach is that in order to prove regularity results on $\mathcal{S}$, we need an explicit construction for it; the abstract completion is not good enough. This leads us to define **Sobolev spaces**, which will turn out that these spaces are the completions we are referring to above (see Chapter 7). We will spend the rest of this chapter and the next on general Sobolev space theory, returning to the study of PDE's in Chapter 7.

5.4 Definition

Suppose $u \in L^1_{\text{loc}}(\Omega)$ and α is any multiindex. A function $v_\alpha \in L^1_{\text{loc}}(\Omega)$ is called the **αth weak derivative of u** if for any $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega v_\alpha\varphi = (-1)^{|\alpha|}\int_\Omega uD^\alpha\varphi.$$

We write $v_\alpha = D^\alpha u$; for this definition to make sense we need to know weak derivatives are unique if they exist.

5.5 Lemma

A αth weak derivative of u, if it exists, is uniquely defined up to a set of measure zero.

◀ Assume $v, w \in L^1_{\text{loc}}(\Omega)$ satisfy

$$\int_\Omega uD^\alpha\varphi = (-1)^{|\alpha|}\int_\Omega v\varphi = (-1)^{|\alpha|}\int_\Omega w\varphi$$

for all $\varphi \in C_c^\infty(\Omega)$. Then $\int_\Omega (v - w)\varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$, and hence $v(x) - w(x) = 0$ for a.e. $x \in \Omega$. ▶

5.6 Definition

A function $u \in L^1_{\text{loc}}(\Omega)$ is called k **times weakly differentiable** if all its weak derivatives exist for $|\alpha| \leq k$. Let $W^k(\Omega)$ denote the set of k times weakly differentiable functions:

$$W^k(\Omega) := \{u \in L^1_{\text{loc}}(\Omega) | u \text{ is } k \text{ times weakly differentiable}\}.$$

Observe that clearly $C^k(\Omega) \subseteq W^k(\Omega)$.

5.7 Examples

1. The function $u : \mathbb{R} \to \mathbb{R}$, $u(x) = |x|$ is weakly differentiable, with

$$Du(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0. \end{cases}$$

2. The function $u : (-1, 1) \to \mathbb{R}$ defined by

$$u(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & -1 \leq x < 0 \end{cases}$$

   is not weakly differentiable, since for $x \neq 0$, u is classically differentiable, with $Du = 0$, and thus as an L^1_{loc}-function, $Du \equiv 0$, and hence Lemma 5.5 would forces the weak derivatives of u to be identically zero. But it is not the case that for every $\varphi \in C_c^\infty((-1, 1))$ that

$$0 = \int_{-1}^{1} \varphi(x) \cdot 0 dx = -\int_{-1}^{1} D\varphi(x)u(x) = -\int_{0}^{1} D\varphi(x)dx = \varphi(0)$$

   (since $\varphi(1) = 0$ as φ is compactly supported in $(-1, 1)$).

5.8 Lemma

Let $\Omega \subseteq \mathbb{R}^n$ be a domain.

1. If $u \in C^0(\Omega)$ and $\sigma > 0$. Then the mollification u_σ converges uniformly to u as $\sigma \to 0$ on any domain $\mathcal{O} \subset\subset \Omega$.
2. If instead $u \in L^p_{\text{loc}}(\Omega)$ (for $p < \infty$) then u_σ converges to u in $L^p_{\text{loc}}(\Omega)$, that is, $\|u_\sigma - u\|_{L^p(\mathcal{O})} \to 0$ as $\sigma \to 0$ for any domain $\mathcal{O} \subset\subset \Omega$.

◀ We have

$$u_\sigma(x) = \sigma^{-n} \int_{B_\sigma(x)} \eta\left(\frac{x-y}{\sigma}\right) u(y)dy = \int_{B_1(0)} \eta(z)u(x - \sigma z)dz,$$

where $z = \frac{x-y}{\sigma}$. Thus if $\mathcal{O} \subset\subset \Omega$ and $2\sigma < \text{dist}(\Omega', \partial\Omega)$ then using the fact that

$$\int_{B_1(0)} \eta = 1,$$

we have

$$\begin{aligned} \sup_{\mathcal{O}} |u - u_\sigma| &\leq \sup_{x \in \mathcal{O}} \int_{B_1(0)} \eta(z)|u(x) - u(x - \sigma z)|dz \\ &\leq \sup_{x \in \mathcal{O}} \sup_{z \in B_1(0)} |u(x) - u(x - \sigma z)|. \end{aligned}$$

Since u is uniformly continuous over the compact set $\{x \in \mathcal{O} \mid \text{dist}(x, \mathcal{O}) \leq \sigma\}$, u_σ tends to u uniformly on $\mathcal{O}$. This proves 1.

To prove 2, since η is non-negative, if $q = 1 - 1/p$ we have by Hölder's inequality

$$\begin{aligned}
|u_\sigma(x)| &= \left|\sigma^{-n}\int_{B_\sigma(x)} \eta\left(\frac{x-y}{\sigma}\right) u(y)dy\right| \\
&\le \int_{B_1(0)} \eta(z)^{\frac{1}{q}} \left(\eta(z)^{\frac{1}{p}} |u(x-\sigma z)|\right) dz \\
&\le \left(\int_{B_1(0)} \eta(z)\right)^{\frac{1}{q}} \left(\int_{B_1(0)} \eta(z)|u(x-\sigma z)|^p dz\right)^{\frac{1}{p}},
\end{aligned}$$

and thus

$$|u_\sigma(x)|^p = \int_{B_1(0)} \eta(z)|u(x-\sigma z)|^p dz.$$

Thus if $\mathcal{O} \subset\subset \Omega$ and $2\sigma < \operatorname{dist}(\mathcal{O}, \partial\Omega)$,

$$\int_{\mathcal{O}} |u_\sigma(x)|^p dx \le \int_{\mathcal{O}}\int_{B_1(0)} \eta(z)|u(x-\sigma z)|^p dz dx,$$

and by Fubini's theorem,

$$\begin{aligned}
\int_{\mathcal{O}}\int_{B_1(0)} \eta(z)|u(x-\sigma z)|^p dz dx &= \int_{B_1(0)} \eta(z) \int_{\mathcal{O}} |u(x-\sigma z)|^p dx dz \\
&\le \int_{B_\sigma(\mathcal{O})} |u(x)|^p dx,
\end{aligned}$$

where $B_\sigma(\mathcal{O}) := \{x \in \Omega \mid \operatorname{dist}(x, \mathcal{O}) < \sigma\}$. Consequently

$$\|u_\sigma\|_{L^p(\mathcal{O})} \le \|u\|_{L^p(B_\sigma(\mathcal{O}))}.$$

Now choose $2\delta < \operatorname{dist}(\mathcal{O}, \partial\Omega)$, $\epsilon > 0$ and a $C^0(\Omega)$ function w satisfying

$$\|u - w\|_{L^p(\mathcal{O})} \le \epsilon$$

(which certainly exists as $C^0(\Omega)$ is dense in $L^p(\Omega)$). Then

$$\|u - u_\sigma\|_{L^p(\mathcal{O})} \le \|u - w\|_{L^p(\mathcal{O})} + \|w - w_\sigma\|_{L^p(\mathcal{O})} + \|w_\sigma - u_\sigma\|_{L^p(\mathcal{O})},$$

and hence by 1,

$$\limsup_{\sigma\to 0} \|u - u_\sigma\|_{L^p(\mathcal{O})} \le 2\epsilon.$$

This completes the proof of 2. ►

5.9 Lemma

Suppose $u \in L^1_{\mathrm{loc}}(\Omega)$ and $D^\alpha u$ exists. Then if $\sigma < \operatorname{dist}(x, \Omega)$,

$$D^\alpha(u_\sigma)(x) = (D^\alpha u)_\sigma(x)$$

(that is, the action of taking weak derivatives commutes with mollification).

◄ Simply compute:

$$\begin{aligned}
(D^\alpha u)_\sigma(x) &= \int_{B_\sigma(x)} \eta_\sigma(x-y) D^\alpha u(y) dy \\
&= (-1)^\alpha \int_{B_\sigma(x)} D_y^\alpha \eta_\sigma(x-y) u(y) dy,
\end{aligned}$$

where $D_y^\alpha \eta_\sigma(x-y)$ is the αth derivative of $\eta_\sigma(x-y)$ as a function of y, where we have used the fact that $\eta_\sigma \in C_c^\infty(\Omega)$. But now by the chain rule

$$(-1)^{|\alpha|} D_x^\alpha \eta_\sigma(x-y) = D_y^\alpha \eta_\sigma(x-y),$$

and thus

$$\begin{aligned}(-1)^{|\alpha|} \int_{B_\sigma(x)} D_y^\alpha \eta_\sigma(x-y)u(y)dy &= \int_{B_\sigma(x)} D_x^\alpha \eta_\sigma(x-y)u(y)dy \\ &= D_x^\alpha \left(\int_{B_\sigma(x)} \eta_\sigma(x-y)u(y)dy \right) \\ &= D^\alpha(u_\sigma)(x). \quad \blacktriangleright\end{aligned}$$

5.10 Proposition

Let Ω be a bounded domain. Let $u, v \in L^1_{\mathrm{loc}}(\Omega)$ and α a multiindex. Then $v = D^\alpha u$ if and only if there exists a sequence $\{w_m\}$ of $C^\infty(\Omega)$ functions converging to u in $L^1_{\mathrm{loc}}(\Omega)$ whose derivative $D^\alpha w_m$ converge to v in $L^1_{\mathrm{loc}}(\Omega)$.

◀ Suppose $v = D^\alpha u$. By Step 1 from the proof of Theorem 2.12, u_σ is smooth. Thus by Lemma 5.8.2 and Lemma 5.9, if we take a sequence $\{\sigma_m\}$ such that $\sigma_m \downarrow 0$ and then set $w_m := u_{\sigma_m}$, we have $w_m \to u$ and $D^\alpha u_m \to v$.

Conversely, suppose $w_m \to u$ and $D^\alpha w_m \to v$ in $L^1_{\mathrm{loc}}(\Omega)$. Take $\varphi \in C_c^\infty(\Omega)$. Then φ and $D^\alpha \varphi$ are bounded, and hence we may apply the dominated convergence theorem to conclude

$$\int_\Omega u D^\alpha \varphi = \lim_{m\to\infty} \int_\Omega u_m D^\alpha \varphi = -\lim_{m\to\infty} \int_\Omega D^\alpha u_m \varphi = -\int_\Omega v\varphi,$$

and thus by the uniqueness result Lemma 5.5, $v = D^\alpha u$. ▶

Note that the converse above actually only required $w_m \in C^{|\alpha|}(\Omega)$.

5.11 Proposition (a chain rule)

Let $f \in C^1(\mathbb{R})$ such that $f' \in L^\infty(\mathbb{R})$. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $u \in W^1(\Omega)$. Then $f \circ u \in W^1(\Omega)$ and $D(f \circ u) = f'(u)Du$.

◀ Choose $\{u_m\}$ to be a sequence of smooth functions in Ω such that $u_m \to u$ and $Du_m \to Du$ in $L^1_{\mathrm{loc}}(\Omega)$. It is enough to show that $f \circ u_m \to f \circ u$ and $f'(u_m) \circ Du_m \to D(f \circ u)$ in $L^1_{\mathrm{loc}}(\Omega)$ (since the previous theorem is an 'if and only if' statement). Take $\mathcal{O} \subset\subset \Omega$. Then since f' is bounded,

$$\int_{\mathcal{O}} |f(u_m) - f(u)| \le \sup_{\mathcal{O}} |f'| \int_{\mathcal{O}} |u_m - u| \to 0 \text{ as } m \to \infty.$$

Next,

$$\begin{aligned}\int_{\mathcal{O}} |f'(u_m)Du_m - f'(u)Du| &\le \int_{\mathcal{O}} |f'(u_m)||Du_m - Du| + \int_{\mathcal{O}} |Du||f'(u_m) - f'(u)| \\ &\le \sup_{\mathcal{O}} |f'| \int_{\mathcal{O}} |Du_m - Du| + \int_{\mathcal{O}} |Du||f'(u_m) - f'(u)|.\end{aligned}$$

The first integral tends to zero as m tends to infinity. By passing to a subsequence if necessary we may assume $u_m(x) \to u(x)$ for a.e. $x \in \mathcal{O}$. Since f' is continuous, $f'(u_m(x)) \to f'(u(x))$ for a.e. $x \in \mathcal{O}$. Since f' is bounded, we have $|f'(u_m) - f'(u)||Du| \le C|Du| \in L^1(\mathcal{O})$, and thus we may apply the dominated convergence theorem to conclude the last integral also tends to zero. This completes the proof. ▶

5.12 Lemma

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, and $u \in W^1(\Omega)$. Then u^+, u^- and $|u|$ are also in $W^1(\Omega)$. Moreover,

$$Du^+(x) = \begin{cases} Du(x) & x > 0 \\ 0 & x \leq 0, \end{cases}$$

$$Du^-(x) = \begin{cases} 0 & x \geq 0 \\ Du(x) & x < 0, \end{cases}$$

$$D|u|(x) = \begin{cases} Du(x) & x > 0 \\ 0 & x = 0 \\ -Du(x) & x < 0. \end{cases}$$

◀ Set

$$f_\epsilon(t) := \begin{cases} \sqrt{t^2+\epsilon^2} - \epsilon & t > 0 \\ 0 & t \leq 0, \end{cases}$$

and observe that $f_\epsilon \in C^1(\mathbb{R})$ and $f'_\epsilon \in L^\infty(\mathbb{R})$; indeed $|f'_\epsilon| \leq 1$ since

$$f'_\epsilon(t) := \begin{cases} \frac{t}{\sqrt{t^2+\epsilon^2}} & t > 0 \\ 0 & t \leq 0. \end{cases}$$

Thus Proposition 5.11 is applicable, and we conclude that $f_\epsilon \circ u \in W^1(\Omega)$ and that moreover for any $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega f_\epsilon(u) D\varphi = -\int_{\{x\in\Omega | u(x)>0|} \varphi \frac{uDu}{\sqrt{u^2+\epsilon^2}},$$

and thus letting $\epsilon \downarrow 0$ (which is applicable on both sides by the dominated convergence theorem) we see that

$$\int_\Omega u^+ D\varphi = \int_{\{x\in\Omega \ | \ u(x)>0\}} \varphi Du,$$

which proves the result for u^+. Since $u^- = -(-u)^+$ and $|u| = u^+ - u^-$, the second and third parts follow immediately from this. ▶

5.13 Corollary

Let $u \in W^1(\Omega)$. Then $Du = 0$ a.e. on any set where u is constant.

◀ Without loss of generality we may assume the constant to be zero. Then the result is immediate from Lemma 5.12, since $Du = Du^+ - Du^-$. ▶

5.14 Definition

Let $\Omega \subseteq \mathbb{R}^n$. Let $1 \leq p < \infty$ and $k \in \mathbb{N}$. Define the **Sobolev space**

$$W^{k,p}(\Omega) := \{u \in W^k(\Omega) | D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq k\}$$

(note that taking $\alpha = (0, \dots, 0)$ shows that $u \in W^{k,p}(\Omega) \Rightarrow u \in L^p(\Omega)$).

Define a norm on $W^{k,p}(\Omega)$ by

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^p \right)^{\frac{1}{p}}.$$

5.15 Lemma

This is indeed a norm.

◀ It is clear that $\|\lambda u\|_{W^{k,p}(\Omega)} = |\lambda| \, \|u\|_{W^{k,p}(\Omega)}$ and that $\|u\|_{W^{k,p}(\Omega)} = 0$ if and only if $u = 0$ a.e., since $\|u\|_{W^{k,p}(\Omega)} \geq \|u\|_{L^p(\Omega)}$. Suppose $u, v \in W^{k,p}(\Omega)$. Then by Minkowski's inequality,

$$\begin{aligned}
\|u+v\|_{W^{k,p}(\Omega)} &= \left(\sum_{|\alpha| \leq k} \|D^\alpha u + D^\alpha v\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} \\
&\leq \left(\sum_{|\alpha| \leq k} \left(\|D^\alpha u\|_{L^p(\Omega)} + \|D^\alpha v\|_{L^p(\Omega)} \right)^p \right)^{\frac{1}{p}} \\
&\leq \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} + \left(\sum_{|\alpha| \leq k} \|D^\alpha v\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} \\
&= \|u\|_{W^{k,p}(\Omega)} + \|v\|_{W^{k,p}(\Omega)} . \quad \blacktriangleright
\end{aligned}$$

5.16 Lemma

The norm

$$\|u\|'_{W^{k,p}(\Omega)} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}$$

is an equivalent norm to $\|\cdot\|_{W^{k,p}(\Omega)}$.

◀ Note that $\|\cdot\|'_{W^{k,p}(\Omega)}$ is clearly a norm. Since the function $f(t) := t^{1/p}$ is convex, we have for any non-negative numbers $n_1, \ldots, n_m$:

$$(n_1^p + \cdots + n_m^p)^{\frac{1}{p}} = f(n_1^p + \cdots + n_m^p) \leq f(n_1^p) + \cdots + f(n_m^p) = n_1 + \cdots + n_m,$$

and thus we have $\|u\|_{W^{k,p}(\Omega)} \leq \|u\|'_{W^{k,p}(\Omega)}$ for all $u \in W^{k,p}(\Omega)$. For the reverse inequality, we observe that there is a function $g(m,p)$ such that there are at most $g(m,p)$ cross terms in the expansion of $(n_1 + \cdots + n_m)^p$, where each n_i is a non-negative number. Since each cross term is at most $n_1^P + \cdots + n_m^p$, we obtain the (poor) estimate

$$(n_1 + \cdots + n_m)^p \leq (1 + g(m,p))(n_1^p + \cdots + n_m^p),$$

and this allows us to conclude $\|u\|'_{W^{k,p}(\Omega)} \leq (1 + g(k,p)) \|u\|_{W^{k,p}(\Omega)}$ for all $u \in W^{k,p}(\Omega)$. The proof is complete. ▶

5.17 Definition

We define $W_0^{k,p}(\Omega) := \overline{C_c^\infty(\Omega)}$ to be the closure of $C_c^\infty(\Omega)$ under the $W^{k,p}(\Omega)$ norm. In general we have $W_0^{k,p}(\Omega) \subsetneq W^{k,p}(\Omega)$.

5.18 Proposition

Both $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ are Banach spaces under the $W^{k,p}(\Omega)$ norm.

◀ It is enough to show that $W^{k,p}(\Omega)$ is complete, as (by definition) $W_0^{k,p}(\Omega)$ is a closed subspace of $W^{k,p}(\Omega)$ and thus inherits the completeness properties of $W^{k,p}(\Omega)$. Assume $\{u_m\} \in W^{k,p}(\Omega)$ is Cauchy under $\|\cdot\|_{W^{k,p}(\Omega)}$. Then for each $|\alpha| \leq k$, $\{D^\alpha u_m\}$ is Cauchy in $L^p(\Omega)$, and thus by completeness of $L^p(\Omega)$ there exists functions $v_\alpha \in L^p(\Omega)$ such that $D^\alpha u_m \to v_\alpha$. In particular,

$u_m \to v_{(0,0\ldots,0)} =: u$ in $L^p(\Omega)$. We now claim that $u \in W^{k,p}(\Omega)$ and in fact $v_\alpha = D^\alpha u$. Indeed, fix $\varphi \in C_c^\infty(\Omega)$ and observe by the dominated convergence theorem,

$$\int_\Omega uD^\alpha\varphi = \lim_{m\to\infty}\int_\Omega u_m D^\alpha\varphi = \lim_{m\to\infty}(-1)^{|\alpha|}\int_\Omega D^\alpha u_m\varphi = (-1)^\alpha\int_\Omega v_\alpha\varphi.$$

Thus our claim is valid. Since $D^\alpha u_m \to D^\alpha u$ in $L^p(\Omega)$ for all $|\alpha| \le k$, we see that $u_m \to u$ in $W^{k,p}(\Omega)$, which completes the proof. ►

5.19 Corollary

$W^{k,2}(\Omega)$ and $W_0^{k,2}(\Omega)$ are Hilbert spaces under the inner product

$$\langle u, v\rangle := \sum_{|\alpha|\le k}\int_\Omega D^\alpha u \cdot D^\alpha v.$$

5.20 Theorem (Poincaré inequality)

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. If $u \in W_0^{1,p}(\Omega)$ then

$$\|u\|_{L^p(\Omega)} \le C\,\|Du\|_{L^p(\Omega)}\,,\ C = C(n,\Omega,p).$$

◄ Note first that convergence in $W^{k,p}(\Omega)$ implies convergence in $L^p(\Omega)$ as $\|u\|_{L^p(\Omega)} \le \|u\|_{W^{k,p}(\Omega)}$. It is therefore enough to prove the result for $u \in C_c^\infty(\Omega)$, as then by choosing $\{u_m\} \in C_c^\infty(\Omega)$ such that $u_m \to u$ in $W_0^{1,p}(\Omega)$ to conclude that

$$\|u\|_{L^p(\Omega)} = \lim_{m\to\infty}\|u_m\|_{L^p(\Omega)} \le C\lim_{m\to\infty}\|Du_m\|_{L^p(\Omega)} = C\,\|Du\|_{L^p(\Omega)}\,.$$

Furthermore, by taking the limit as $p \downarrow 1$, we may assume $p > 1$.

So let $p > 1$, and $u \in C_c^\infty(\Omega)$. Fix a point $y \in \Omega$, and let X be the C^1 vector field

$$X(x) = |u(x)|^p(x-y)$$

(this is C^1 as $p > 1$). Then

$$\mathrm{div}_x(X) = n|u|^p + p|u|^{p-1}\mathrm{sgn}(u)Du\cdot(x-y),$$

where the subscript 'x' indicates that we are taking the divergence of X as a function of x, and

$$\mathrm{sgn}(u)(x) = \begin{cases}1 & u(x) \ge 0\\ 0 & u(x) = 0\\ -1 & u(x) < 0.\end{cases}$$

Since u is compactly supported, the divergence theorem gives us $\int_\Omega \mathrm{div}_x(X) = 0$, and hence

$$n\int_\Omega |u|^p = -p\int_\Omega |u|^{p-1}\mathrm{sgn}(u)Du\cdot(x-y).$$

Thus

$$\int_\Omega |u|^p \le \frac{pd}{n}\int_\Omega |u|^{p-1}|Du|,$$

where $d := \mathrm{diam}(\Omega)$. We now apply Hölder's inequality to conclude that

$$\int_\Omega |u|^p \le \frac{pd}{n}\left(\int_\Omega |u|^{(p-1)q}\right)^{\frac{1}{q}}\left(\int_\Omega |Du|^p\right)^{\frac{1}{p}},$$

and then dividing through by $\left(\int_\Omega |u|^{(p-1)q}\right)^{1/q} = \left(\int_\Omega |u|^p\right)^{1-1/p}$ we obtain

$$\left(\int_\Omega |u|^p\right)^{\frac{1}{p}} \le \frac{pd}{n}\left(\int_\Omega |Du|^p\right)^{\frac{1}{p}}.$$

This completes the proof. ►

5.21 Corollary

Let $\Omega \subseteq \mathbb{R}^n$ be bounded. In $W_0^{k,p}(\Omega)$ the norm

$$\|u\|_{W_0^{k,p}(\Omega)} := \sum_{|\alpha|=k} \|D^\alpha u\|_{L^p(\Omega)}$$

is an equivalent norm to the norm $\|\cdot\|_{W^{k,p}(\Omega)}$.

It is enough to show that there exists $C \geq 0$ such that $\|u\|_{W^{k,p}(\Omega)} \leq C \|u\|_{W_0^{k,p}(\Omega)}$ for all $u \in W_0^{k,p}(\Omega)$ (the converse is trivially true). We in fact show that there exists $C \geq 0$ such that $\|u\|'_{W^{k,p}(\Omega)} \leq C \|u\|_{W_0^{k,p}(\Omega)}$ for all $u \in W_0^{k,p}(\Omega)$, which suffices by Lemma 5.16. This however is immediate by induction and the previous theorem. ►

5.22 Lemma

If $\varphi \in C_c^\infty(\Omega)$ and $u \in W^{k,p}(\Omega)$ then $\varphi u \in W^{k,p}(\Omega)$ and

$$D^\alpha(\varphi u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \varphi D^{\alpha-\beta} u,$$

where $\beta \leq \alpha$ if and only if $\beta_i \leq \alpha_i$ for all i, and

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}.$$

◄ Induction on $|\alpha|$. Suppose $|\alpha| = 1$. Choose any $\psi \in C_c^\infty(\Omega)$. Then as $D^\alpha(\varphi\psi) = \varphi D^\alpha \psi + \psi D^\alpha \varphi$,

$$\int_\Omega \varphi u D^\alpha \psi = \int_\Omega (u D^\alpha(\varphi\psi) - u(D^\alpha\varphi)\psi = -\int (\varphi D^\alpha u + u D^\alpha \varphi)\psi,$$

and hence $D^\alpha(\varphi u) = \varphi D^\alpha u + u D^\alpha \varphi$ as desired.

Now suppose $\ell < k$, and the desired result holds for all $|\alpha| \leq \ell$ and for all $\varphi \in C_c^\infty(\Omega)$. Choose a multiindex α with $|\alpha| = \ell + 1$. Then $\alpha = \beta + \gamma$, where $|\beta| = \ell$ and $|\gamma| = 1$. Then with ψ as above,

$$\int_\Omega \varphi u D^\alpha \psi = \int_\Omega \varphi u D^\beta(D^\gamma \psi) = (-1)^{|\beta|} \int_\Omega \sum_{\varepsilon \leq \beta} \binom{\beta}{\varepsilon} D^\varepsilon \varphi D^{\beta-\varepsilon} u D^\gamma \psi$$

(by the inductive assumption applied to $D^\gamma \psi \in C_c^\infty(\Omega)$)

$$= (-1)^{|\beta|+|\gamma|} \int_\Omega \sum_{\varepsilon \leq \beta} \binom{\beta}{\varepsilon} D^\gamma (D^\varepsilon \varphi D^{\beta-\varepsilon} u)\psi$$

(by the induction assumption again, this time with 'u'$= D^{\beta-\varepsilon}u \in W^{k-|\beta|+|\varepsilon|,p}(\Omega)$, and $D^\varepsilon \varphi \in C_c^\infty(\Omega)$)

$$= (-1)^{|\alpha|} \int_\Omega \sum_{\varepsilon \leq \beta} \binom{\beta}{\varepsilon} \left(D^\varepsilon \varphi D^{\alpha-\varepsilon} u + D^\tau \varphi D^{\alpha-\tau} u\right) \psi$$

(where $\tau = \varepsilon + \gamma$, so $\alpha - \tau = \beta - \varepsilon$)

$$= (-1)^{|\alpha|} \int_\Omega \left(\sum_{0 \leq \varepsilon \leq \beta} \binom{\alpha-\gamma}{\varepsilon} D^\varepsilon \varphi D^{\alpha-\varepsilon} u + \sum_{\gamma \leq \tau \leq \alpha} \binom{\alpha-\gamma}{\tau-\gamma} D^\tau \varphi D^{\alpha-\tau} u \right) \psi$$

$$= (-1)^{|\alpha|} \int_\Omega \left(\sum_{0 \leq \varepsilon \leq \alpha} \binom{\alpha}{\varepsilon} D^\varepsilon \varphi D^{\alpha-\varepsilon} u \right) \psi,$$

since
$$\binom{\alpha-\varepsilon}{\varepsilon-\gamma}+\binom{\alpha-\varepsilon}{\varepsilon}=\binom{\alpha}{\varepsilon}.$$
This completes the proof of the induction step, and thus of the lemma. ►

This result is a minor extension of Proposition 5.10.

5.23 Lemma (local approximation by smooth functions)

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, and let $u \in W^{k,p}(\Omega)$, where $1 \leq p < \infty$. Then $u_\sigma \to u$ in $W^{k,p}_{\mathrm{loc}}(\Omega)$ as $\sigma \to 0$.

◄ By Lemma 5.8.2 and Lemma 5.9, together with the definition of Sobolev spaces, we have $D^\alpha u_\sigma \to D^\alpha u$ in $L^p(\mathcal{O})$ for any $\mathcal{O} \subset\subset \Omega$ and any $|\alpha| \leq k$. Consequently,
$$\|u_\alpha - u\|^p_{W^{k,p}(\mathcal{O})} = \sum_{|\alpha|\leq k} \|D^\alpha u_\sigma - D^\alpha u\|^p_{L^p(\mathcal{O})} \to 0$$
as $\sigma \to 0$. This completes the proof. ►

We can now use this lemma to derive an important density theorem for $W^{k,p}(\Omega)$.

5.24 Theorem (global approximation by smooth functions)

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. Then $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

◄ Let $\{\Omega_i \mid i \in \mathbb{N}\}$ be a nested sequence of subdomains of Ω, with $\Omega_i \subset\subset \Omega_{i+1}$ and $\bigcup_{i\in\mathbb{N}} \Omega_i = \Omega$. Set $\Omega_0 = \Omega_{-1} = \emptyset$, and then set $U_i := \Omega_{i+1} \backslash \Omega_{i-1}$, for $i \geq 0$. Let $\{\rho_i\}_{i\geq 0}$ be a partition of unity subordinate to the open cover $\{U_i\}_{i\geq 0}$ of Ω.

Now given $u \in W^{k,p}(\Omega)$ and $\epsilon > 0$, choose $\{\sigma_i\}$ positive real numbers such that
$$\sigma_i \leq \operatorname{dist}(\Omega_i, \partial\Omega_{i+1}),$$
and
$$\left\|(\rho_i u)_{\sigma_i} - \rho_i u\right\|_{W^{k,p}(\Omega)} \leq \frac{\epsilon}{2^i},$$
where $(\rho_i u)_{\sigma_i}$ is the mollification of $\rho_i u$ - note that $(\rho_i u)_{\sigma_i}$ is supported in U_i and that Lemma 5.22 shows that $\rho_i u \in W^{k,p}(\Omega)$, and hence Lemma 5.23 justifies the existence of such $\{\sigma_i\}$.

Now set
$$v := \sum_{i=1}^\infty (\rho_i u)_{\sigma_i}.$$
Observe that on any subdomain $\mathcal{O} \subset\subset \Omega$, v is a finite sum, and hence $v \in C^\infty(\Omega)$. Since $\sum_{i=1}^\infty \rho_i \equiv 1$, we have for any $\mathcal{O} \subset\subset \Omega$,
$$\|v-u\|_{W^{k,p}(\mathcal{O})} \leq \sum_{i=1}^\infty \left\|(\rho_i u)_{\sigma_i} - \rho_i u\right\| \leq \sum_{i=1}^\infty \frac{\epsilon}{2^i} = \epsilon.$$
Now we can take the supremum over $\mathcal{O} \subset\subset \Omega$ to conclude that $\|v-u\|_{W^{k,p}(\Omega)} \leq \epsilon$, and this completes the proof. ►

It is **not** true in general that if $\Omega \subseteq \mathbb{R}^n$ is a bounded domain then $C^\infty\left(\bar{\Omega}\right) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$. However if we assume additional regularity properties on Ω, we obtain the following result, which we won't prove.

5.25 Theorem

Let $\Omega \subseteq \mathbb{R}^n$ is a bounded domain, such that $\partial\Omega$ is Lipschitz then $C^\infty\left(\bar{\Omega}\right) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

6 Embedding theorems

In this chapter we state and prove important embedding theorems. More precisely, we wish to find $\ell = \ell(k,p,n)$ such that $W^{k,p}(\Omega) \subseteq C^{\ell}(\Omega)$ for a domain $\Omega \subseteq \mathbb{R}^n$. We then discuss when the embedding is compact; this will be extremely important in Chapter 7.

6.1 Theorem (Sobolev embedding theorems)

Let $\Omega \subseteq \mathbb{R}^n$ be a domain.

1. **(Sobolev's inequality)** If $1 \leq p < n$, set $p^* := \frac{np}{n-p}$, so $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. Then there exists a constant $C = C(n,p)$ such that for all $u \in W_0^{1,p}(\Omega)$,
$$\|u\|_{L^{p^*}(\Omega)} \leq C \|Du\|_{L^p(\Omega)},$$
and hence $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$.
2. **(Morrey's inequality)** If $n < p < \infty$, set $\gamma = 1 - \frac{n}{p}$. Then exists a constant $C = C(n,p)$ such that for all $u \in W_0^{1,p}(\Omega)$,
$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} \leq C \|Du\|_{L^p(\Omega)},$$
and hence $W_0^{1,p}(\Omega) \hookrightarrow C^{0,\gamma}(\overline{\Omega})$ (where $\|u\|_{C^{0,\gamma}(\Omega)} = \sup_{\Omega}|u| + \sup_{x\neq y\in\Omega}\frac{|u(x)-u(y)|}{|x-y|^{\gamma}}$ - see Definition 6.6).

This is arguably the hardest result in the course, and will take us some time to prove. Moreover it is not the most general result of this kind; we shall prove a more general embedding theorem in Theorem 6.14, and in Section 6.17 we discuss the extension to the case of $W^{k,p}(\Omega)$ instead of $W_0^{k,p}(\Omega)$.

6.2 The case $p = n$

We will also say a quick word about the **Sobolev borderline case** $p = n$. In this case one can prove:

- If Ω is a **bounded** domain then for any $q \in [n,\infty)$, there exists a constant $C = C(n,q)$ such that for all $u \in W_0^{1,n}(\Omega)$,
$$\|u\|_{L^n(\Omega)} \leq C|\Omega|^{\frac{1}{q}} \|Du\|_{L^q(\Omega)},$$
and hence $W_0^{1,n}(\Omega) \hookrightarrow L^q(\Omega)$.

This will be less important for us (although we will need it in the proof of Theorem 6.14), and we will not prove it.

6.3 Dimension counting

Before we embark on the proof of Theorem 6.1 let us make the following remark. Suppose we knew that Sobolev's inequality (Theorem 6.1.1) held for some p^*. It turns out that by 'dimension counting' we can work out what p^* must be. Here is the argument.

Take $1 \leq p < n$ and suppose p^* is such that
$$\|u\|_{L^{p^*}(\Omega)} \leq C \|Du\|_{L^o(\Omega)}. \tag{7}$$
Set $u_\lambda(x) := u(\lambda x)$. Then
$$\int_\Omega |u_\lambda(x)|^{p^*}\,dx = \int_\Omega |u(\lambda x)|^{p^*}\,dx = \frac{1}{\lambda^n}\int_\Omega |u(x)|^{p^*}\,dx$$

and
$$\int_\Omega |Du_\lambda(x)|^p\,dx = \int_\Omega |Du(\lambda x)|^p\,dx = \frac{\lambda^p}{\lambda^n}\int_\Omega |Du(x)|^p\,dx,$$
and hence by (7),
$$\lambda^{-\frac{n}{p^*}}\|u\|_{L^{p^*}(\Omega)} \le C\lambda\cdot\lambda^{-\frac{n}{p}}\|Du\|_{L^p(\Omega)},$$
and hence
$$\|u\|_{L^{p^*}(\Omega)} \le C\lambda^{1-\frac{n}{p}-\frac{n}{p^*}}\|Du\|_{L^p(\Omega)}.$$
But now if $1-\frac{n}{p}-\frac{n}{p^*}\neq 0$ then by either letting $\lambda\to 0$ or $\lambda\to\infty$ we obtain a contradiction. Thus for Theorem 6.1.1 to hold for some p^* we must have
$$1-\frac{n}{p}-\frac{n}{p^*}=0,$$
that is,
$$p^*=\frac{np}{n-p}.$$

6.4 Generalised Holder inequality

To prove Theorem 6.1 we will need the **generalised Holder inequality**, which states that if $1\le p_1\le\cdots\le p_m\le\infty$ are real numbers such that $\sum_{i=1}^m \frac{1}{p_i}=1$, then if $u_k\in L^{p_k}(\Omega)$ for $k=1,\ldots,m$ we have
$$\int_\Omega |u_1\ldots u_m| \le \prod_{k=1}^m \|u_k\|_{L^{p_k}(\Omega)}.$$

6.5 Proof of Sobolev's inequality (Theorem 6.1.1)

◀ We will first prove the estimate for $u\in C_c^\infty(\Omega)$, that is, we show for $u\in C_c^\infty(\Omega)$ we have
$$\|u\|_{L^{p^*}(\Omega)} \le C\|Du\|_{L^p(\Omega)}.$$

Assume first that $p=1$, so $p^*=\frac{n}{n-1}$. Since u has compact support, extending u to be identically zero outside Ω we have for each $i=1,\ldots,n$ and $x\in\Omega$ that
$$u(x)=\int_{-\infty}^{x_i} D_i u(x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n)dy_i,$$
and so
$$|u(x)|^{\frac{n}{n-1}} \le \left(\prod_{i=1}^n \int_{-\infty}^{\infty}|Du|dy_i\right)^{\frac{1}{n-1}}.$$
Thus integrating the above with respect to x_1 gives
$$\begin{aligned}
\int_{-\infty}^{\infty}|u|^{\frac{n}{n-1}}dx_1 &\le \int_{-\infty}^{\infty}\prod_{i=1}^n\left(\int_{-\infty}^{\infty}|Du|dy_i\right)^{\frac{1}{n-1}}dx_1\\
&\le \left(\int_{-\infty}^{\infty}|Du|dy_1\right)^{\frac{1}{n-1}}\int_{-\infty}^{\infty}\prod_{i=2}^n\left(\int_{-\infty}^{\infty}|Du|dy_i\right)^{\frac{1}{n-1}}dx_1\\
&\le \left(\int_{-\infty}^{\infty}|Du|dy_1\right)^{\frac{1}{n-1}}\left(\prod_{i=2}^n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|dx_1dy_i\right)^{\frac{1}{n-1}},
\end{aligned}$$
where we have used the generalised Holder inequality, with each $p_i=\frac{1}{n-1}$.

Now we repeat and integrate with respect to x_2 to obtain
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u|^{\frac{n}{n-1}}dx_1dx_2 \le \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|dy_1dy_2\right)^{\frac{1}{n-1}}\int_{-\infty}^{\infty}\prod_{i\neq 2} I_i^{\frac{1}{n-1}}dx_2,$$

where
$$I_1 := \int_{-\infty}^{\infty} |Du| dy_1$$
and
$$I_i := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du| dx_1 dy_i \text{ for } i = 3, \dots, n.$$
Applying the generalised Holder inequaity again we see that have $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1 dx_2$ is less than or equal to
$$\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du| dx_1 dy_2\right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du| dy_1 dx_2\right)^{\frac{1}{n-1}} \prod_{i=3}^{n} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du| dx_1 dx_2 dy_i\right)^{\frac{1}{n-1}}.$$
Continuing in this way we obtain
$$\int_\Omega |u|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |Du| dx_1 \dots dy_i \dots dy_n\right)^{\frac{1}{n-1}} = \left(\int_\Omega |Du|\right)^{\frac{n}{n-1}},$$
which proves the estimate for $p = 1$ (here $C(n, p) = 1$).

Now suppose $1 < p < n$, and consider $|u|^\gamma$ for some constant $\gamma > 0$. Then since $|u|^\gamma \in W_0^{1,1}(\Omega)$, from the case $p = 1$, we have
$$\begin{aligned} \left(\int_\Omega |u|^{\frac{\gamma n}{n-1}}\right)^{\frac{n-1}{n}} &\leq \int_\Omega |D|u|^\gamma| \\ &= \gamma \int_\Omega |u|^{\gamma-1}|Du| \\ &\leq \gamma \left(\int_\Omega |u|^{\frac{(\gamma-1)p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int_\Omega |Du|^p\right)^{\frac{1}{p}}, \end{aligned}$$
where we used Hölder's inequality on the last line.

Now set
$$\gamma := \frac{p(n-1)}{n-p} > 1,$$
so $\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1} = p^* = \frac{np}{n-p}$. Then we obtain
$$\left(\int_\Omega |u|^{p^*}\right)^{\frac{\gamma}{p^*}} \leq \gamma \left(\int_\Omega |u|^{p^*}\right)^{\frac{\gamma-1}{p^*}} \left(\int_\Omega |Du|^p\right)^{\frac{1}{p}},$$
and thus
$$\|u\|_{L^{p^*}(\Omega)} \leq \frac{p(n-1)}{n-p} \|Du\|_{L^p(\Omega)},$$
so $C(n, p) = \frac{p(n-1)}{n-p}$.

This proves the stated estimate for $u \in C_c^\infty(\Omega)$. It remain to prove the estimate for arbitrary $u \in W_0^{1,p}(\Omega)$ and show that $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$. For this, pick $u \in W_0^{1,p}(\Omega)$ and let $\{u_m\}$ be a sequence of $C_c^\infty(\Omega)$ functions converging to u in $W^{1,p}(\Omega)$. Applying the estimates to the differences $u_m - u_{m'}$, we see that $\{u_m\}$ is a Cauchy sequence is $L^{p^*}(\Omega)$. Thus $u_m \to \tilde{u} \in L^{p^*}(\Omega)$ say. But then we have $u(x) = \lim_{m\to\infty} u_m(x) = \tilde{u}(x)$ for a.e. $x \in \Omega$, and hence $u = \tilde{u}$ (up to a set of measure zero), which proves $W_0^{1,p}(\Omega)$ embeds in $L^{p^*}(\Omega)$ as required.

Finally, since we have
$$\|u_m\|_{L^{p^*}(\Omega)} \leq C \|Du_m\|_{L^p(\Omega)},$$
since $u_m \to u = \tilde{u}$ in both $W_0^{1,p}(\Omega)$ and $L^{p^*}(\Omega)$, we also have
$$\|u\|_{L^{p^*}(\Omega)} \leq C \|Du\|_{L^p(\Omega)}.$$
The proof is complete. ▶

We now aim to prove Morrey's inequality (Theorem 6.1.2). The key ingredient will be the Theorem 6.7 below, also due to Morrey.

6.6 Definition

Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $\gamma \in (0,1)$. Recall that we define the **γ-Hölder** norm for a function $u \in C^0(\Omega)$ by

$$\|u\|_{C^{0,\gamma}(\Omega)} := \sup_{\Omega} |u| + \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x-y|^\gamma}$$

(which is not necessarily finite) and let $C^{0,\gamma}(\Omega)$ denote the set of $u \in C^0(\Omega)$ such that $\|u\|_{C^{0,\gamma}(\Omega)} < \infty$.

6.7 Theorem (Morrey's Lemma)

Let $n < p \leq \infty$. If $u \in C^1(\mathbb{R}^n)$ then there exists a constant $C = C(n,p)$ such that if $\gamma = 1 - \frac{n}{p}$ then

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

Note that we are not neccesarily assuming the right-hand side is finite, although if it isn't, the result is trivial, so we may assume that the right-hand side is finite. Observe we are using the full Sobolev norm on the right-side, since u is not assumed to be compactly supported, so Lemma 5.16 is not applicable.

◀ We will break the proof down into 3 stages.

Step 1: The key estimate: choose any ball $B_\rho(x) \subseteq \mathbb{R}^n$. We will show there exists a constant $C = C(n)$, such that

$$\int_{B_\rho(x)} |u(y) - u(x)| dy \leq \frac{\rho^n}{n} \int_{B_\rho(x)} \frac{|Du(y)|}{|y-x|^{n-1}} dy. \tag{8}$$

To prove this, fix any point $\xi \in \partial B_1(0)$. Then if $0 < s < \rho$, we have

$$\begin{aligned} |u(x+s\xi) - u(x)| &= \left| \int_0^1 \frac{d}{dt}(u(x+st\xi) dt \right| \\ &\leq s \int_0^1 |Du(x+st\xi)| dt, \end{aligned}$$

since $|\xi| = 1$. Thus integrating over $\partial B_1(0)$ and changing the order of integration, we obtain

$$\int_{\partial B_1(0)} |u(x+s\xi) - u(x)| dS(\xi) \leq s \int_0^1 \int_{\partial B_1(0)} |Du(x - st\xi)| dS(\xi) dt.$$

Now we change variables to $y = x + s\xi$ on the left-hand side and $z = x + st\xi$ on the right-hand side to obtain

$$\begin{aligned} s^{1-n} \int_{\partial B_s(x)} |u(y) - u(x)| dy &\leq s \int_0^1 (st)^{1-n} \int_{\partial B_{st}(x)} |Du(z)| dz dt \\ &\leq s \int_0^1 \int_{\partial B_{st}(x)} \frac{|Du(z)|}{|z-x|^{n-1}} dz dt \quad \text{since } st = |z-x| \text{ when } z \in \partial B_{sr}(x), \\ &\leq \int_0^s \int_{\partial B_\sigma(x)} \frac{|Du(z)|}{|z-x|^{n-1}} dz d\sigma \quad \text{with } \sigma = st, \\ &= \int_{B_s(x)} \frac{|Du(z)|}{|z-x|^{n-1}} dz. \end{aligned}$$

So far we therefore have

$$s^{1-n} \int_{\partial B_s(x)} |u(y) - u(x)| dy \leq \int_{B_s(x)} \frac{|Du(z)|}{|z-x|^{n-1}} dz,$$

and so multiplying both sides by s^{n-1} and integrating s from 0 to ρ we obtain

$$\int_0^\rho \int_{\partial B_s(x)} |u(y)-u(x)|dyds \leq \int_0^\rho s^{n-1} \int_{B_s(x)} \frac{|Du(z)|}{|z-x|^{n-1}} dzds,$$

which gives

$$\int_{B_\rho(x)} |u(y)-u(x)|dy \leq \frac{\rho^n}{n} \int_{B_\rho(x)} \frac{|Du(z)|}{|z-x|^{n-1}} dz,$$

which is equation 8.

Step 2: Estimating $\sup |u|$. Now we show that

$$\sup_{\mathbb{R}^n} |u| \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}, \tag{9}$$

where $C = C(n,p)$.

We have

$$|u(x)| \leq |u(x)-u(y)| + |u(y)|,$$

and so integrating over $B_1(x)$ with respect to y,

$$|u(x)|\omega_n \leq \int_{B_1(x)} |u(x)-u(y)|dy + \int_{B_1(x)} |u(y)|dy.$$

Now taking $\rho = 1$ in (8) and applying Hölder's inequality to the second term, we obtain

$$|u(x)|\omega_n \leq \frac{1}{n} \int_{B_1(0)} \frac{|Du(y)|}{|y-x|^{n-1}} dy + \left(\int_{B_1(x)} dy \right)^{\frac{p-1}{p}} \|u\|_{L^p(\mathbb{R}^n)}.$$

Now we apply Hölder's inequality to the first integral to obtain

$$\begin{aligned} |u(x)|\omega_n &\leq \frac{1}{n} \left(\int_{B_1(0)} |y-x|^{\frac{p(1-n)}{p-1}} dy \right)^{\frac{p-1}{p}} \|Du\|_{L^p(\mathbb{R}^n)} + \left(\int_{B_1(x)} dy \right)^{\frac{p-1}{p}} \|u\|_{L^p(\mathbb{R}^n)} \\ &\leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}. \end{aligned}$$

The last equality holds since $p > n$ implies $\frac{p(1-n)}{p-1} > -n$, and thus $\int_{B_1(0)} |y-x|^{\frac{p(1-n)}{p-1}} dy$ is finite.

Step 3: Estimating the Holder constant. Now we prove that with $\gamma = 1 - \frac{n}{p}$, there exists $C = C(n,p)$ such that for $x \neq y \in \mathbb{R}^n$,

$$\frac{|u(x)-u(y)|}{|x-y|^\gamma} \leq C \|Du\|_{L^p(\Omega)}. \tag{10}$$

Once we have shown this will have completed the proof, since combining (9) and (10) we will have the existence of a constant $C = C(n,p)$ such that

$$\sup_{\mathbb{R}^n} |u| + \sup_{x \neq y \in \mathbb{R}^n} \frac{|u(x)-u(y)|}{|x-y|^\gamma} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

So fix $x \neq y$ and set $\rho = |x-y|$. Let $W := B_\rho(x) \cap B_\rho(y)$. Then we have

$$|u(x)-u(y)| \leq |u(x)-u(z)| + |u(z)-u(y)|,$$

and so integrating over W with respect to z gives

$$\begin{aligned}
|u(x)-u(y)|C\rho^n &\leq \int_W |u(x)-u(z)|dz + \int_W |u(z)-u(y)|dz \\
&\leq \int_{B_\rho(x)} |u(x)-u(z)dz + \int_{B_\rho(y)} |u(z)-u(y)dz \quad \text{since } W \subseteq B_\rho(x), W \subseteq B_\rho(y), \\
&\overset{(*)}{\leq} \frac{\rho^n}{n}\left(\int_{B_\rho(x)} \frac{|Du(z)|}{|z-x|^{n-1}}dz + \int_{B_\rho(y)} \frac{|Du(z)|}{|z-y|^{n-1}}dz\right) \\
&\overset{(**)}{\leq} \frac{\rho^n}{n}\left(\int_{B_\rho(x)} |x-z|^{\frac{p(1-n)}{p-1}}dz\right)^{\frac{p-1}{p}} \|Du\|_{L^p(B_\rho(x))} \\
&\quad +\frac{\rho^n}{n}\left(\int_{B_\rho(y)} |y-z|^{\frac{p(1-n)}{p-1}}dz\right)^{\frac{p-1}{p}} \|Du\|_{L^p(B_\rho(y))},
\end{aligned}$$

where $(*)$ used (8) and $(**)$ is by Hölder's inequality.

This time we are forced to actually compute $\int_{B_\rho(x)} |x-z|^{\frac{p(1-n)}{p-1}}dz$.

Let $r = |x-z|$. We obtain

$$\int_0^\rho r^{(n-1)+\frac{p(1-n)}{p-1}}dr = C\left(\rho^{n+\frac{p(1-n)}{p-1}}\right),$$

where $C = C(n,p)$, and thus

$$(**) = \frac{\rho^n}{n}\cdot C\rho^{\frac{n(p-1)}{p}+(1-n)}\left(\|Du\|_{L^p(B_\rho(x))} + \|Du\|_{L^p(B_\rho(y))}\right) = C\rho^n\rho^\gamma\left(\|Du\|_{L^p(B_\rho(x))} + \|Du\|_{L^p(B_\rho(y))}\right),$$

and taking the norm over the larger domain of $\mathbb{R}^n$ finally gives us

$$|u(x)-u(y)| \leq C\rho^\gamma \|Du\|_{L^p(\mathbb{R}^n)}.$$

Recalling that $\rho = |x-y|$, this gives the desired estimate and thus completes the proof of Morrey's lemma. ▶

6.8 Proof of Morrey's inequality (Theorem 6.1.2)

◀ The proof of Morrey's inequality is now very straightforward, that is, that if $n < p < \infty$, then if $\gamma = 1 - \frac{n}{p}$, there exists a constant $C = C(n,p)$ such that for all $u \in W_0^{1,p}(\Omega)$,

$$\|u\|_{C^{0,\gamma}(\bar{\Omega})} \leq C\|Du\|_{L^p(\Omega)},$$

and hence $W_0^{1,p}(\Omega) \hookrightarrow C^{0,\gamma}\left(\bar{\Omega}\right)$.

Indeed, given $u \in W_0^{1,p}(\Omega)$, let $\{u_m\}$ be a sequence of $C_c^\infty(\Omega)$ functions converging to u in $W^{1,p}(\Omega)$. Then by Morrey's lemma (Theorem 6.7), we have $\sup_\Omega |u_m|$ bounded for all m (by $\|u\|_{W^{1,p}(\Omega)}$), so $\{u_m\}$ is uniformly bounded. Similary, as for all m and all $x, y \in \Omega$, we have $|u_m(x) - u_m(y)| \leq |x-y|^\gamma$, the sequence $\{u_m\}$ is equicontinuous. Hence by Arzela-Ascoli, passing to a subsequence if neccessary, we may assume $u_m \to \tilde{u}$ uniformly on compacta, where $\tilde{u} \in C^{0,\gamma}(\Omega)$. But then we have $u(x) = \lim_{m\to\infty} u_m(x) = \tilde{u}(x)$ for a.e. $x \in \Omega$, and hence $u = \tilde{u}$ (up to a set of measure zero), which proves $W_0^{1,p}(\Omega)$ embeds in $C^{0,\gamma}\left(\bar{\Omega}\right)$ as required.

Finally, since we have

$$\|u_m\|_{C^{0,\gamma}(\bar{\Omega})} \leq C\|u_m\|_{W^{1,p}(\Omega)},$$

since $u_m \to u = \tilde{u}$ in both $W_0^{1,p}(\Omega)$ and $C^{0,\gamma}\left(\bar{\Omega}\right)$, we also have

$$\|u\|_{C^{0,\gamma}(\bar{\Omega})} \leq C\|u\|_{W^{1,p}(\Omega)}.$$

The proof is complete. ▶

This thus finally completes the proof of the Sobolev embedding theorems. We will now gradually move onto tackling the issue of compactness; the statement we are aiming for is **Rellich's Compactness Theorem**, given in Theorem 6.16 below.

6.9 Corollary

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. If $n < p < \infty$ then any $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a.e. (classically) differentiable, and its classical derivative agrees with its weak derivative.

◀ First we need a slight variant on the estimate (8). Suppose $v \in C^1(\Omega)$ and $B_{2\rho}(x) \subset\subset \Omega$. Then if $y \in B_\rho(x)$, we claim

$$|v(x) - v(y)| \leq C\rho^{1-\frac{n}{p}} \|Dv\|_{L^p(B_{2\rho}(x))}\,. \tag{11}$$

The proof proceeds almost identically to that of (8):

$$|v(x) - v(y)| \leq |v(x) - v(z)| + |v(z) - v(y)|,$$

and then integrating over $B_\rho(y)$ gives

$$\omega_n\rho^n|v(x) - v(y)| \leq \int_{B_\rho(y)} |v(x) - v(z)|dz + \int_{B_\rho(y)} |v(z) - v(y)|dy.$$

Then if in the first integral we integrate over the larger domain $B_{2\rho}(x)$, we can see as in the proof of (8) that

$$\int_{B_\rho(y)} |v(x)-v(z)|dz+\int_{B_\rho(y)} |v(z)-v(y)|dy \leq \frac{(2\rho)^n}{n}\int_{B_{2\rho}(x)} \frac{|Dv(z)|}{|z-x|^{n-1}}dz+\frac{\rho^n}{n}\int_{B_\rho(y)}\frac{|Du(z)|}{|z-y|^{n-1}}dz,$$

and then as in the proof of (8) this provides the desired result.

Next, since $C^1(\Omega)\cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$ by Theorem 5.24 (this is why we require $p < \infty$ in the statement of this corollary), this estimate remains true for $v \in W^{1,p}(\Omega)$ by approximation; moreover since we integrate only over balls compactly contained in Ω, this also holds throughout $W^{1,p}_{\mathrm{loc}}(\Omega)$.

Now let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$. Then for a.e. $x \in \Omega$, we have by Lebesgue's differentiation theorem (see Section 1.3),

$$\frac{1}{\omega_n\rho^n}\int_{B_\rho(x)} |Du(x) - Du(z)|dx \to 0 \text{ as } \rho \downarrow 0$$

(more precisely, for every Lebesgue point $x \in \Omega$ this holds, and a.e. x is a Lebesgue point).

Fix any such point x, and set

$$v(y) := u(y) - u(x) - Du(x) \cdot (y - x).$$

Note that $v(x) = 0$, and $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$, with $Dv(y) = Du(y) - Du(x)$. We apply the estimate above, with $\rho = |x - y|$ to obtain

$$|u(y) - u(x) - Du(x) \cdot (y - x)| \leq C\rho\left(\int_{B_{2\rho}(x)} |Du(x) - Du(z)|^p dz\right)^{\frac{1}{p}} = o(\rho) = o(|x - y|),$$

by Lebesgue's differentiation theorem.

This proves u is a.e. classically differentiable, and its gradient is precisely the weak gradient. ▶

6.10 Definition

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. Let $u \in L^1_{\mathrm{loc}}(\Omega)$. Given $h \in \mathbb{R}$ and $\mathcal{O} \subset\subset \Omega$ such that $\mathrm{dist}(\mathcal{O}, \partial\Omega) > |h|$, define the **$j$th difference quotient of size h of u** to be

$$\Delta^h_j u(x) := \frac{u(x + he_j) - u(x)}{h},$$

and set $\Delta^h u = (\Delta^h_1 u, \dots, \Delta^h_n u)$.

6.11 Proposition (first difference quotient lemma)

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, and let $\mathcal{O} \subset\subset \Omega$, and $0 < |h| < \operatorname{dist}(\mathcal{O}, \partial\Omega)$. Let $u \in W^{1,p}(\Omega)$, where $1 \leq p < \infty$. Then $\Delta_j^h u \in L^p(\mathcal{O})$ and

$$\left\|\Delta_j^h u\right\|_{L^p(\mathcal{O})} \leq \|D_j u\|_{L^p(\Omega)}.$$

◀ First suppose $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$. Then

$$\Delta_j^h u(x) = \frac{u(x+he_j) - u(x)}{h} = \frac{1}{h}\int_0^h D_j(x_1, \ldots, x_{i-1}, x_i + t, x_{i+1}, \ldots, x_n)dt,$$

and thus by Hölder's inequality,

$$|\Delta_j^h u(x)|^p \leq \frac{1}{h}\int_0^h |D_j(x_1, \ldots, x_{i-1}, x_i + t, x_{i+1}, \ldots, x_n)|^p dt,$$

and thus by Fubini,

$$\int_{\mathcal{O}} |\Delta_j^h u|^p dx \leq \frac{1}{h}\int_0^h \int_{B_{|h|}(\mathcal{O})} |D_j u|^p dx dt \leq \int_\Omega |D_j u|^p dx,$$

where $B_{|h|}(\mathcal{O}) = \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, \mathcal{O}) < |h|\} \subseteq \Omega$.

Then since $C^1(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$, given $u \in W^{1,p}(\Omega)$ we can find $\{u_m\} \in C^1(\Omega) \cap W^{1,p}(\Omega)$ such that $u_m \to u$ in $W^{1,p}(\Omega)$. Then $\left\{\Delta_j^h u_m\right\}$ is Cauchy in $L^p(\mathcal{O})$, and thus converges to some $\Delta_j^h \tilde{u}$, say. But of course, $\lim_{m\to\infty} \Delta_j^h u_m = \Delta_j^h u$, and so $\Delta_j^h u \in L^p(\mathcal{O})$, and then finally as

$$\left\|\Delta_j^h u\right\|_{L^p(\Omega)} \leq \|D_j u_m\|_{L^p(\Omega)}$$

for all m, passing to the limit in m completes the proof. ▶

6.12 Weak convergence

We recall the concept of **weak convergence**. Let X denote a real Banach space. We say a subsequence $\{u_m\}$ in X converges **weakly** to $u \in X$, written $u_m \rightharpoonup u$, if for each $f \in X^*$ we have $f(u_m) \to f(u)$ in $\mathbb{R}$.

Convergence is easily seen to imply weak convergence, and any weakly convergent subsequence is bounded. Moreover, if $u_m \rightharpoonup u$ then

$$\|u\| \leq \liminf_{m\to\infty} \|u_m\|.$$

If X is a **reflexive** Banach space and $\{u_m\}$ is a bounded sequence, then there exists a subsequence (m') and $u \in X$ such that $\{u_{m'}\}$ is weakly convergent to u. In particular, if $\{u_m\}$ is a bounded sequence in $L^p(\Omega)$ (for $\Omega \subseteq \mathbb{R}^n$ open and $1 < p < \infty$), then there exists a subsequence (m') and $u \in L^p(\Omega)$ such for all $v \in L^q(\Omega)$ (where $\frac{1}{p} + \frac{1}{q} = 1$) we have

$$\int_\Omega v u_m \to \int_\Omega v u$$

(this uses the fact that $L^p(\Omega)^* \cong L^q(\Omega)$, and so any bounded linear functional f on $L^p(\Omega)$ can be written as $u \overset{f}{\mapsto} \int_\Omega vu$ for some $v \in L^q(\Omega)$), and $\|u\|_{L^p(\Omega)} \leq \liminf_{m\to\infty} \|u_m\|_{L^p(\Omega)}$).

6.13 Proposition (second difference quotient lemma)

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. Let $u \in L^p(\Omega)$, where $1 < p < \infty$. Suppose there exists a constant K such that $\Delta_j^h u \in L^p(\mathcal{O})$ and $\left\|\Delta_j^h u\right\|_{L^p(\mathcal{O})} \leq K$ for all $h > 0$ and $\mathcal{O} \subset\subset \Omega$ satisfying $0 < h < \operatorname{dist}(\mathcal{O}, \partial\Omega)$. Then the weak derivative $D_j u$ exists and satisfies

$$\|D_j u\|_{L^p(\Omega)} \leq K.$$

◀ Choose any sequence $\{h_m\} \downarrow 0$. Set $u_m := \Delta_j^{h_m} u$. Let $\mathcal{O} \subset\subset \Omega$ be arbitrary, Then for m sufficiently large we have

$$\|u_m\|_{L^p(\mathcal{O})} \leq K.$$

Thus there exists a subseqence $\{m'\}$ and $v \in L^p(\mathcal{O})$ such that $u_{m'} \rightharpoonup v$, with $\|v\|_{L^p(\mathcal{O})} \leq K$. In particular, for any $\varphi \in C_c^1(\Omega)$ we have

$$\int_{\mathcal{O}} \varphi u_{m'} \to \int_{\mathcal{O}} \varphi v.$$

Now for $h_m < \operatorname{dist}(\operatorname{supp}(\varphi), \partial\Omega)$, we have

$$\begin{aligned} \int_{\mathcal{O}} \varphi u_{m'} &= \int_{\mathcal{O}} \left(\frac{u(x + h_{m'} e_j) - u(x)}{h} \right) \varphi(x) dx \\ &= \frac{1}{h} \int_{\mathcal{O}} u(y)\varphi(y - h_{m'} e_j) - u(y)\varphi(y) dy \\ &= \int_{\mathcal{O}} u(y) \left(\frac{\varphi(y - h_{m'} e_j) - \varphi(y)}{h} \right) dy \\ &= -\int_{\mathcal{O}} u \Delta_j^{-h_{m'}} \varphi \end{aligned} \tag{12}$$

(note we do not need to change domains here as φ is compactly supported).

But now by the dominated convergence theorem,

$$-\int_{\mathcal{O}} u \Delta_j^{-h_{m'}} \varphi \to -\int_{\mathcal{O}} u D_j \varphi.$$

Thus $v = D_j u$ and so $D_j u \in L^p(\mathcal{O})$. Since $\mathcal{O}$ and j were arbitrary, we conclude $u \in W^{1,p}(\Omega)$. Finally, since

$$\|v\|_{L^p(\mathcal{O})} \leq \liminf_{m \to \infty} \|u_m\|_{L^p(\mathcal{O})} \leq K,$$

we have shown $\|D_j u\|_{L^p(\mathcal{O})} \leq K$ for all $\Omega' \subset\subset \Omega$, we conclude that $\|D_j u\|_{L^p(\Omega)} \leq K$. The proof is complete. ▶

We record here for use later that in (12) we proved the following **integral identity** for difference quotients: that

$$\int_{\Omega} v \Delta_j^h u = -\int_{\Omega} u \Delta_j^{-h} v. \tag{13}$$

Here is the extension to Theorem 6.1 to the case where k is not necessarily equal to 1.

6.14 Theorem (general Sobolev inequalities)

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open domain. Let $u \in W_0^{k,p}(\Omega)$. Then:

1. If $k < \frac{n}{p}$ then $u \in L^{p^*}(\Omega)$ where $p^* = \frac{np}{n-kp}$ (so $\frac{1}{p^*} = \frac{1}{p} - \frac{k}{n}$). In addition we have the estimate

$$\|u\|_{L^{p^*}(\Omega)} \leq C \|u\|_{W_0^{k,p}(\Omega)},$$

   where $C = C(k, n, p, \Omega)$ (where the $\|\cdot\|_{W_0^{k,p}(\Omega)}$ is the norm from Corollary 5.21).

2. If $k > \frac{n}{p}$ then $u \in C^{m,\gamma}\left(\bar{\Omega}\right)$, where $m = k - \left\lfloor \frac{n}{p} \right\rfloor - 1$ and

$$\gamma = \begin{cases} 1 + \left\lfloor \frac{n}{p} \right\rfloor - \frac{n}{p} & \frac{n}{p} \notin \mathbb{N} \\ \text{any number } \gamma \text{ such that } 0 < \gamma < 1 & \frac{n}{p} \in \mathbb{N}. \end{cases}$$

   In addition we have the estimate

$$\|u\|_{C^{m,\gamma}(\overline{\Omega})} \leq C \|u\|_{W_0^{k,p}(\Omega)},$$

   where $C = C(k, p, n, \gamma, \Omega)$.

◀ First assume the hypotheses of 1. Then $D^\alpha u \in L^p(\Omega)$ for all $|\alpha| = k$, and we have for any β such that $|\beta| = k-1$ that $D^\beta u \in L^{p_1}(\Omega)$, and $D^\beta u \in W_0^{1,p}(\Omega)$. Hence by the Sobolev inequality (Theorem 6.1.1) we have $D^\beta u \in L^{p_1}(\Omega)$ where $\frac{1}{p_1} = \frac{1}{p} - \frac{1}{n}$, and moreover that

$$\left\|D^\beta u\right\|_{L^{p_1}(\Omega)} \leq C \left\|D^\alpha u\right\|_{L^p(\Omega)} \leq C \left\|u\right\|_{W_0^{k,p}(\Omega)}.$$

Thus $u \in W_0^{k-1,p_1}(\Omega)$. Similarly we find that $u \in W_0^{k-2,p_2}(\Omega)$ where $\frac{1}{p_2} = \frac{1}{p_1} - \frac{1}{n} = \frac{1}{p} - \frac{2}{n}$. After k steps we have $u \in W_0^{0,p^*}(\Omega) = L^{p^*}(\Omega)$, where p^* is an in the statement. The stated estimate is obtained by multiplying the relevant estimates at each stage of the above argument. This proves 1.

To prove 2, first suppose $\frac{n}{p} \notin \mathbb{N}$. Then as before we see that $u \in W_0^{k-\ell,r}(\Omega)$ for $\frac{1}{r} = \frac{1}{p} - \frac{\ell}{n}$, provided $\ell p < n$. Choose ℓ such that

$$\ell < \frac{n}{p} < \ell + 1;$$

that is, $\ell = \left\lfloor \frac{n}{p} \right\rfloor$. Then $r = \frac{np}{n - \ell p} > n$. Hence by Morrey's inequality (Theorem 6.1.2) we have $D^\alpha u \in C^{0,1-\frac{n}{r}}(\bar{\Omega})$ for all $|\alpha| \leq k - \ell - 1$. Observe that

$$1 - \frac{n}{r} = 1 - \frac{n}{p} - \ell = 1 + \left\lfloor \frac{n}{p} \right\rfloor - \frac{n}{p} = \gamma,$$

and hence $u \in C^{m,\gamma}\left(\bar{\Omega}\right)$, with $m = k - \ell - 1 = k - \left\lfloor \frac{n}{p} \right\rfloor - 1$. The stated estimate follows by multiplying, as before. This proves 13 when $\frac{n}{p} \notin \mathbb{N}$.

Finally, suppose $\frac{n}{p} \in \mathbb{N}$. Set $\ell = \frac{n}{p} - 1$. Then as above $u \in W_0^{k-\ell,r}(\Omega)$ where $r = \frac{np}{n-\ell p} = n$. Then $D^\alpha u \in W_0^{1,n}(\Omega)$ for any $|\alpha| \leq k - \ell - 1 = k - \frac{n}{p}$. Thus the (unproved) assertion for the Sobolev borderline case (see Section 6.2) shows that $D^\alpha u \in L^q(\Omega)$ for any $n \leq q < \infty$. Then Morrey's inequality (Theorem 6.1.2) shows that $D^\alpha u \in C^{0,1-\frac{n}{q}}\left(\bar{\Omega}\right)$ for all $n < q < \infty$ and all $|\alpha| \leq k - \left\lfloor \frac{n}{p} \right\rfloor - 1 = m$. Thus $u \in C^{m,\gamma}\left(\bar{\Omega}\right)$ for an $0 < \gamma < 1$. As before, the stated estimate follows by multiplying, and this completes the proof. ▶

All the embeddings we have proved so far are necessarily continuous, as the embedding operator $\mathcal{I} : X \hookrightarrow Y$ (where $X = W_0^{k,p}(\Omega)$ and Y is whichever space corresponds to these particular values of k, n, p) is a bounded linear functional (since we always proved bounds) and hence is continuous. For applications to PDE however, we will need to know more that just that $\mathcal{I}$ is continuous. We require compactness, in the following sense.

6.15 Definition

Let $X \subseteq Y$ be Banach spaces, and $\mathcal{I} : X \hookrightarrow Y$ the embedding operator. We say that X is **compactly embedded** in Y, written $X \subset\subset Y$, if $\mathcal{I}$ is a compact operator, that is, for any bounded set $A \subseteq X$ we have $\mathcal{I}(A)$ precompact in Y.

6.16 Theorem (Rellich's compactness theorem)

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. Suppose $u \in W_0^{k,p}(\Omega)$.

1. If $kp < n$, set $p^* = \frac{np}{n-kp}$. Then for any $1 \leq q < p^*$ we have $W_0^{k,p}(\Omega) \subset\subset L^q(\Omega)$.

2. If $kp > n$, then $W_0^{k,p}(\Omega) \subset\subset C^{m,\beta}\left(\bar{\Omega}\right)$,where β is any number such that $0 < \beta < \gamma$, and m, γ are as in the statement of Theorem 6.14.2.

3. If $kp = n$, then $W_0^{k,p}(\Omega) \subset\subset L^q(\Omega)$ for any $1 \leq q < \infty$.

We won't prove 3.

◀ First we prove 1, which is by far the hardest part. Let $A \subseteq W_0^{k,p}(\Omega)$ be bounded, so say for all $u \in A$ we have

$$\|u\|_{W_0^{k,p}(\Omega)} \leq K.$$

We want to show that $\overline{A}^{\|\cdot\|_{L^q(\Omega)}}$ is compact in $L^q(\Omega)$, for any $1 \leq q < p^*$, where the notation $\overline{A}^{\|\cdot\|_{L^q(\Omega)}}$ indicates we are taking the closure with respect to the $L^q(\Omega)$ norm. Since we are working in metric spaces, it is enough to show that A is totally bounded as a subset of $L^q(\Omega)$.

We prove this first in the special case that $q = 1$. First we will show that $A_\sigma := \{u_\sigma \mid u \in A\}$ is totally bounded. Note first that in the following calculations, we do not need to bother with alterning the domain to $\Omega_\sigma = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \sigma\}$ as any $u \in A$ is compactly supported in Ω. We have for any $u \in A$ that for any $x \in \Omega$,

$$|u_\sigma(x)| = \left|\sigma^{-n} \int_{B_\sigma(x)} \eta\left(\frac{x-y}{\sigma}\right) u(y)dy\right| \leq \sigma^{-n} \sup \eta \, \|u\|_{L^1(\Omega)},$$

and moreover

$$|Du_\sigma(x)| = \left|\sigma^{-n-1} \int_{B_\sigma(x)} D\eta\left(\frac{x-y}{\sigma}\right) u(y)dy\right| \leq \sigma^{-n-1} \sup |D\eta| \, \|u\|_{L^1(\Omega)}.$$

Thus A_σ is a uniformly bounded, equicontinuous subset of $C^0\left(\bar{\Omega}\right)$, and hence precompact in $C^0\left(\bar{\Omega}\right)$ by the Arzela-Ascoli theorem, and thus also precompact in $L^1(\Omega)$.

Next, we estimate for $u \in A \cap C_c^\infty(\Omega)$ that

$$\begin{aligned}
|u(x) - u_\sigma(x)| &\leq \int_{B_1(0)} \eta(z)|u(x) - u(x - \sigma z)|dz \\
&= \int_{B_1(0)} \eta(z) \left|\int_0^1 \frac{d}{dt}(u(x - t\sigma z)dt\right| dz \\
&= \sigma \int_{B_1(0)} \eta(z) \left|\int_0^1 Du(x - t\sigma z) \cdot z dt\right| dz \\
&\leq \sigma \int_{B_1(0)} \int_0^1 \eta(z)|Du(x - t\sigma z)|dtdz,
\end{aligned}$$

and thus integrating over Ω (if necessary, we could integrate over $\mathbb{R}^n$ instead since u is compactly supported, and thus we can extend u by defining u to be zero off Ω) we obtain

$$\begin{aligned}
\int_\Omega |u(x) - u_\sigma(u)dx &\leq \sigma \int_{B_1(0)} \eta(z) \int_0^1 \int_\Omega |Du(x - t\sigma z)dxdtdz \\
&\leq \sigma \int_\Omega \|Du\|_{L^1(\Omega)} \\
&= \sigma|\Omega| \, \|Du\|_{L^1(\Omega)}.
\end{aligned}$$

Hence since A is bounded in $W^{k,p}(\Omega)$ there exists a constant C such that

$$\|u_\sigma - u\|_{L^1(\Omega)} \leq C\sigma,$$

and hence A is contained in a neighborhood of A_σ, that is, $A \subseteq B_{C\sigma}(A_\sigma)$ (where the neighborhood is with respect to the $L^1(\Omega)$ norm), and consequently A is totally bounded in $L^1(\Omega)$. This establishes the result for $q = 1$. Note that we have not yet used the embedding theorems.

Now take $1 < q < p^*$. Observe that

$$\begin{aligned}
\int_\Omega |u|^q &= \int_\Omega |u|^{(1-\lambda)q}|u|^{\lambda q} \\
&\leq \left(\int_\Omega |u|^{p_1((1-\lambda)q)}\right)^{\frac{1}{p_1}} \left(\int_\Omega |u|^{p_2\lambda q}\right)^{\frac{1}{p_2}}
\end{aligned}$$

for any $0 < \lambda < 1$ and any $p_1, p_2 \geq 1$ such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$.

Since $1 < q < p^*$ there exists $0 < \lambda < 1$ such that

$$\frac{p^*}{(1-\lambda)q} + \frac{1}{\lambda q} = 1.$$

Set $p_1 = \frac{p^*}{(1-\lambda)q}$ and $p_2 = \frac{1}{\lambda q}$. Then we obtain

$$\int_\Omega |u|^q \leq \left(\int_\Omega |u|^{p^*}\right)^{\frac{(1-\lambda)q}{p^*}} \left(\int_\Omega |u|\right)^{\lambda q},$$

(where the right-hand side is well defined by the Sobolev embedding theorem), and thus

$$\|u\|_{L^q(\Omega)} \leq \|u\|_{L^{p^*}(\Omega)}^{1-\lambda} \|u\|_{L^1(\Omega)} \leq C \|u\|_{W^{k,p}(\Omega)}^{1-\lambda} \|u\|_{L^1(\Omega)} \leq C' \|u\|_{L^1(\Omega)}.$$

Since A is totally bounded in $L^1(\Omega)$ it follows A is totally bounded in $L^q(\Omega)$, and this proves 1.

To prove 2, which is much easier, if $kp > n$, fix $0 < \beta < \gamma$. Then if A is a bounded set in the $W^{k,p}(\Omega)$ norm of functions in $W_0^{k,p}(\Omega)$ then A is bounded and equicontinuous under the $C^{m,\beta}\left(\bar{\Omega}\right)$ norm and hence by Arzela-Ascoli is precompact. The proof is complete. ▶

6.17 Extension to $W^{k,p}(\Omega)$

The various embedding and compactness theorems can be extended to $W^{k,p}(\Omega)$ (instead of just $W_0^{k,p}(\Omega)$) if we assume some additional regularity on the boundary of Ω. This is fairly easy to prove if Ω has C^1 boundary, and remains true if Ω has only Lipschitz boundary.

Finally we remark that the various embedding and compactness theorems we have proved are optimal - given values of k, n, p the associated values p^*, m, γ are the best possible.

7 Weak solutions of the Dirichlet problem

In this chapter we will show how Sobolev spaces allow to us obtain **weak solutions** to PDE's, and thus connect the technical definition of Sobolev spaces with the original motivation we gave in Section 5.1.

7.1 Assumptions

Until further notice, we make the following assumptions:

- $\Omega \subseteq \mathbb{R}^n$ is a bounded domain.
- $Lu = D_i(a_{ij}D_j u) + b_j D_j u + cu$ is a **divergence form** operator satisfying:
  - **(E) uniform ellipticity**: there exists $\lambda > 0$ such that for all $x \in \Omega$ and all $\xi \in \mathbb{R}^n$ we have $a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$,
  - **(B) boundedness**: a_{ij}, b_j, c are measurable functions such that there exist $\Lambda, M > 0$ such that for all $x \in \Omega$,

$$\sum_{i,j=1}^n |a_{ij}(x)|^2 \leq \Lambda,$$

$$\lambda^{-2}\sum_{j=1}^n |b_j(x)|^2 + \lambda^{-1}|c(x)|^2 \leq M.$$

7.2 Definitions

Let $\varphi \in W^{1,2}(\Omega)$, $g, f_1, \ldots, f_n \in L^2(\Omega)$. Define:

$$A(u,v) := \int_\Omega \sum_{i,j} a_{ij} D_j u D_i v - \sum_j b_j (D_j u) v - cu,$$

$$F(v) := \int_\Omega \sum_i f_i D_i v - gv.$$

We consider the Dirichlet Problem**:** $(\bigstar)$

$$Lu = g + \sum_i D_i f_i \quad \text{in} \quad \Omega, \quad u = \varphi \quad \text{on} \quad \partial\Omega$$

for $u \in W^{1,2}(\Omega)$.

Note that this problem doesn't appear well posed, as f_i is not necessarily (weakly) differentiable, and since $\partial\Omega$ is a set of measure zero, it is meaningless to require $u = \varphi$ on $\partial\Omega$ since u is only defined up to a set of measure zero!

Here is how we get round this: we say $u \in W^{1,2}(\Omega)$ is a **weak solution** to $(\bigstar)$ if for all $v \in W_0^{1,2}(\Omega)$ we have

$$A(u,v) = F(v)$$

and that also

$$u - \varphi \in W_0^{1,2}(\Omega).$$

Note that these **do** make sense!

The next step is to formulate and prove a weak maximum prinicple for (weak) solutions to $(\bigstar)$. In order for this to be possible we need to be able to make sense of expressions like $u \leq 0$ on $\partial\Omega$ for $u \in W^{1,2}(\Omega)$. That is the content of the next definition.

7.3 Definitions

Given $u \in W^{1,2}(\Omega)$ we say that $u \leq 0$ on $\partial\Omega$ if $u^+ \in W_0^{1,2}(\Omega)$ (note $u^+ \in W^{1,2}(\Omega)$ by Lemma5.12). Observe that is $u \in W^{1,2}(\Omega) \cap C^0(\Omega)$ then $u^+ \in W_0^{1,2}(\Omega)$ if and only if $u(x) \leq 0$ for all $x \in \partial\Omega$. Similarly we say $u \geq 0$ on $\partial\Omega$ if $-u \leq 0$ on $\partial\Omega$, and we say $u \leq v$ on $\partial\Omega$ if $u - v \leq 0$ on $\partial\Omega$.

Next, we define for $u \in W^{1,2}(\Omega)$

$$\sup_{\partial\Omega} u := \inf\{k \mid u \leq k \text{ on } \partial\Omega\},$$

and similarly

$$\inf_{\partial\Omega} u := -\sup_{\partial\Omega}(-u).$$

With these definitions on board, here is the version of the weak maximum prinicple we wish to prove.

7.4 Theorem (Weak maximum prinicple)

Suppose u is a **subsolution** of L, that is, $Lu \geq 0$ on $\partial\Omega$. Then if $c(x) \leq 0$ for all $x \in \Omega$,

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+,$$

where the supremum on the left is the **essential supremum**.

◀ Since $c \leq 0$ the assumption $Lu \geq 0$ is equivalent to the statement that for all non-negative $v \in W_0^{1,2}(\Omega)$ we have $A(u,v) \geq 0$.

Suppose the result is false, and choose t such that $\sup_{\partial\Omega} u < t < \sup_\Omega u$. Set $v := (u(x) - t)^+$ - note that $v \in W_0^{1,2}(\Omega)$ and v is non-negative and

$$Dv = \begin{cases} Du & u > t \\ 0 & u \leq t. \end{cases}$$

Then

$$A(u, v) = \int_{\Omega \cap \{u>t\}} a_{ij} D_j v D_i v - b_j (D_j v) v - cu(u - t) \leq 0,$$

and hence using uniform ellipticity **(E)** and the fact that c is non-negative, we obtain

$$\begin{aligned} \lambda \int_{\Omega\cap\{u>t\}} |Dv|^2 &\leq \int_{\Omega\cap\{u>t\}} b_j (D_j v) v \\ &\leq \lambda\sqrt{M} \left(\int_{\Omega\cap\{u>t\}} |Dv|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega\cap\{u>t\}} v^2 \right)^{\frac{1}{2}} \end{aligned}$$

(by Hölder's inequality and **(B)**).

Now if we set $\Gamma := \mathrm{supp}(Dv) \subseteq \mathrm{supp}(v)$, we may as well write instead

$$\int_\Omega |Dv|^2 \leq M \int_\Gamma v^2,$$

or

$$\|Dv\|_{L^2(\Omega)}^2 \leq M \|v\|_{L^2(\Gamma)}^2$$

Now we apply Theorem 6.1.1 to obtain for $n \geq 3$ that

$$C^{-1} \|v\|_{L^{\frac{2n}{n-2}}(\Omega)} \leq \|Dv\|_{L^2(\Omega)}.$$

But then by Hölder's inequality,

$$\begin{aligned} \int_\Gamma v^2 &\leq \left(\int_\Gamma v^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \left(\int_\Gamma dx \right)^{1-\frac{n-2}{n}} \\ &\leq \|v\|^2_{L^{\frac{2n}{n-2}}(\Omega)} |\Gamma|^{\frac{2}{n}}, \end{aligned}$$

(where $|\Gamma|$ is the Lebesgue volume of Γ) and thus we obtain

$$C^{-1} \|v\|_{L^{\frac{2n}{n-2}}(\Omega)} \leq M^{\frac{1}{2}} \|v\|_{L^{\frac{2n}{n-2}}(\Omega)} |\Gamma|^{\frac{1}{n}}.$$

In other words, there exists a constant $K = K(n, M)$ such that

$$|\Gamma| \geq K^{-n}.$$

But this inequality is independent of t, and thus holds as $t \to \sup_\Omega u$. Hence u attains its supremum on a set of positive measure, but then on Γ we have $Du = 0$ by Corollary 5.13, contradicting the choice of Γ. ▶

7.5 Corollary (uniqueness of solutions to (★))

Let $u, v \in W_0^{1,2}(\Omega)$ be two (weak) solutions of (★). Then $u \equiv v$.

◀ It is enough to note by the weak maximum principle above and the corresponding weak minimum prinicple that if $Lu = 0$ in Ω then $u = 0$ in Ω. ▶

We now need some functional analysis to proceed further with our study of weak solutions.

7.6 Theorem (Lax-Milgram Theorem)

Let H be a Hilbert space and $B : H \times H \to H$ a bilinear map such that:

- B is **bounded**, that is, there exists $K \geq 0$ such that $|B(x, y)| \leq K \|x\| \|y\|$ for all $x, y \in H$,
- B is **coercive**, that is, there exists $\nu \geq 0$ such that $B(x, x) \geq \nu \|x\|^2$ for all $x \in H$.

Then if $F \in H^*$ is any bounded linear functional there exists a unique $w_F \in H$ such that

$$F(x) = B(w_F, x) \text{ for all } x \in H.$$

Note that if B is symmetric then this follows immediately from the Riesz representation theorem, as we may define a new inner product $\langle \cdot, \cdot \rangle_B$ on H by setting

$$\langle x, y \rangle_B := B(x, y),$$

and then applying the Reisz representation theorem to $\langle \cdot, \cdot \rangle_B$.

◀ Given $x \in H$, consider the element $f_x \in H^*$ defined by $f_x(y) = B(x, y)$. By the Riesz representation theorem there exists a unique $z_x \in H$ such that

$$f_x(y) = \langle z_x, y \rangle \text{ for all } y \in H.$$

Define $T : H \to H$ by $Tx = z_x$, so

$$B(x, y) = \langle Tx, y \rangle .$$

Then T is linear, and moreover since

$$\|Tx\|^2 = \langle Tx, Tx \rangle = A(x, Tx) \leq K \|x\| \|Tx\| ,$$
$$\|Tx\| \|x\| \geq \langle Tx, x \rangle = A(x, x) \geq \nu \|x\|^2 ,$$

we obtain $\nu \leq \|T\| \leq K$ and thus T is bounded and bounded below.

Since T is bounded below, T is injective. Moreover the range of T, $R(T)$ is closed: if $\{Tx_n\}$ is Cauchy, then so is $\{x_n\}$, since

$$\|Tx_n - Tx_m\| \geq \nu \|x_n - x_m\| \to 0,$$

and thus if $x_n \to x$ then $Tx_n \to Tx \in R(T)$. Next, if $y \in R(T)^\perp$ then since

$$\nu \|y\| \leq B(y, y) = \langle Ty, y \rangle = 0,$$

we have $y = 0$, and consequently T is a bounded linear isomorphism, with inverse T^{-1}.

Now given $F \in H^*$, apply the Riesz representation to find $z_F \in H$ such that $F(x) = \langle z_F, x \rangle$, and then set $w_F := T^{-1} z_F$. Then

$$F(x) = \langle z_F, x \rangle = \langle Tw_F, x \rangle = A(w_F, x),$$

and the proof is complete. ▶

We recall without proof the **Fredholm alternative**:

7.7 Theorem (Fredholm alternative)

Let $T : H \to H$ be a compact linear operator on a Hilbert space H. Then precisely one of the following holds:

1. the homogeneous equation $x - Tx = 0$ has a non-trivial solution $x \in H$,
2. for every $y \in H$ the equation $x - Tx = y$ has a uniquely determined solution $x \in H$.

Moreover in the second case, the operator $(I - T)^{-1}$ whose existence is asserted there is bounded.

Using these results we can now prove the key result of this chapter.

7.8 Theorem (Solving (★))

If $c \le 0$ in Ω the problem (★) is uniquely solvable for any choice of $g, f_1, \dots, f_n \in L^2(\Omega)$ and any $\varphi \in W^{1,2}(\Omega)$.

◄ Set $H := W_0^{1,2}(\Omega)$ for this proof. We proceed in six steps.

Step 1: Reduction to zero boundary data: setting $w = u - \varphi$, we see that

$$A(w,v) = A(u,v) - A(\varphi, v),$$

and thus u solves (★) if and only if $A(w,v) = G(v)$ for all $v \in H$ where

$$\begin{aligned} G(v) &= F(v) - A(\varphi, v) \\ &= \int_\Omega \sum_{i=1}^n \left(\sum_{j=1}^n f_j - a_{ij} D_j \varphi \right) D_i v + \left(g - c\varphi - \sum_{j=1}^n b_j D_j \varphi \right) v \\ &=: \int_\Omega \sum_{i=1}^n \tilde{f}_i D_i v + \tilde{g} v; \end{aligned}$$

note that $\tilde{g}, \tilde{f}_1, \dots, \tilde{f}_n \in L^2(\Omega)$ and $w \in H$. Thus we have reduced (★) to the zero boundary data problem.

Step 2: Showing that A is almost coercive: observe that

$$\begin{aligned} A(u,u) &= \int_\Omega a_{ij} D_j u D_i u - b_j (D_j u) u - cu \\ &\ge \int_\Omega \lambda |Du|^2 - b_j (D_j u) u - cu. \end{aligned}$$

Using the fact that $ab \le \epsilon a^2 + \frac{1}{4\epsilon} b^2$ for any $a, b, \epsilon > 0$ (which follows from writing

$$ab = (a\sqrt{2\epsilon}) \left(\frac{b}{\sqrt{2\epsilon}} \right),$$

and applying Cauchy-Schwarz), we see that

$$\int_\Omega |Du||u| \le \epsilon \int_\Omega |Du|^2 + \frac{1}{4\epsilon} \int_\Omega u^2.$$

Thus if we choose $\epsilon = \lambda/2$, we see that

$$\begin{aligned} \int_\Omega \sum_{j=1}^n (D_j u) b_j u &\le \frac{\lambda}{2} \int_\Omega |Du|^2 + \frac{\lambda}{2} \frac{1}{\lambda^2} \sup_{x \in \Omega} \left(\sum_{j=1}^n |b_j(x)|^2 \right) \int_\Omega u^2 \\ &= \frac{\lambda}{2} \int_\Omega |Du|^2 + \frac{\lambda M}{2} \int_\Omega u^2, \end{aligned}$$

and thus

$$A(u,u) \ge \frac{\lambda}{2} \int_\Omega |Du|^2 - \frac{\lambda M}{2} \int_\Omega u^2.$$

In other words, A is almost coercive.

Step 3: Modifying L to make A coercive: define $L_\sigma u := Lu - \sigma u$. Then the corresponding bilinear form A_σ satisfies

$$A_\sigma(u,u) = A(u,u) + \sigma \int_\Omega u^2.$$

Now choose σ such that $\sigma \geq \lambda M$. Then we obtain

$$\begin{aligned} A_\sigma(u,u) &\geq \frac{\lambda}{2}\left(\int_\Omega |Du|^2 + \int_\Omega u^2\right) \\ &= \frac{\lambda}{2}\|u\|^2_{W^{1,2}(\Omega)}, \end{aligned}$$

and thus A_σ is coercive.

Step 4: Applying the Lax-Milgram Theorem. We claim that A_σ is bounded. It is enough to check that A is bounded, and this follows as

$$\begin{aligned} |A(u,v)| &= \left|\int_\Omega a_{ij}D_juD_iv - b_j(D_ju)v - cuv\right| \\ &\leq \sum_{i,j=1}^n \sup_{x\in\Omega}|a_{ij}(x)|\int_\Omega |Du||Dv| + \sum_{i=1}^n \sup_{x\in\Omega}|b_j(x)|\int_\Omega |Du||v| + \sup_{x\in\Omega}|c(x)|\int_\Omega |u||v| \\ &\leq K\|u\|_{W^{1,2}(\Omega)}\|v\|_{W^{1,2}(\Omega)} \end{aligned}$$

for some $K \geq 0$.

Now we claim that $F(v) = \int_\Omega \sum_{i=1}^n f_iD_iv + gv$ is a bounded linear functional on H. This is clear, since the f_i and g are in $L^2(\Omega)$. Thus we may apply the Lax-Milgram Theorem 7.6 to obtain a unique $w \in H$ such that

$$A_\sigma(w,v) = \langle L_\sigma w, v\rangle = F(v)$$

for all $v \in H$.

Step 5: Using the Fredholm alternative. Now note that:

$$\begin{aligned} Lu = F &\Leftrightarrow L_\sigma u + \sigma Ju = F \\ &\Leftrightarrow u + \sigma L_\sigma^{-1}(Ju) = L_\sigma^{-1}(F), \end{aligned}$$

where J is the embedding $H \to H^*$ defined by

$$Ju(v) = \int_\Omega uv.$$

Set $Tu = -\sigma L_\sigma^{-1}(Ju)$. If we knew that T was compact, since the equation

$$u - Tu = 0$$

has no non-trivial solutions by Corollary 7.5 (as $u = Tu$ if and only if $Lu = 0$), we could infer by theFredholm alternative (Theorem 7.7) that the equation

$$u - Tu = v$$

was uniquely solvable for all $v \in H$, and thus that the equation $Lu = F$ was uniqely solvable for all $F \in H^*$. This would therefore complete the proof.

Step 6: It thus remains to show that T is compact. Since L_σ^{-1} is continuous, it is enough to show that J is compact. To prove this we write J as the composition $J = J_1 \circ J_2$ where $J_2 : H \hookrightarrow L^2(\Omega)$ and $J_1 : L^2(\Omega) \to H^*$ is given by

$$J_1u(v) = \int_\Omega uv.$$

Since J_1 is clearly continuous, it is enough to show that J_2 is compact. But this is precisely the statement of Theorem 6.16 (we either use Theorem 6.16.2 or Theorem 6.16.3 depending as to whether $n > 2$ or $n = 2$ respectively). Hence J_2 is compact, thus so is T; this completes the proof. ▶

8 Regularity of weak solutions

We have proved the existence and uniqueness of weak solutions to ($\bigstar$) under certain conditions. Unfortunately a weak solution is in general not much good if that is all it is; the aim now is to show that (under extra conditions) the weak solutions are actually bona fide real solutions to the problem at hand. The first step is improve $W^{1,2}$-regularity to $W^{2,2}$-regularity.

8.1 Theorem (interior $W^{2,2}(\Omega)$ regularity)

Let $u \in W^{1,2}(\Omega)$ be a weak solution of the equation $Lu = f$, where L is uniformly elliptic in the bounded domain $\Omega \subseteq \mathbb{R}^n$,

$$Lu = D_i(a_{ij}D_ju) + b_jD_ju + cu,$$

where the $a_{ij} \in C^{0,1}\left(\bar{\Omega}\right)$, the b_j and c are in $L^\infty(\Omega)$ and $f \in L^2(\Omega)$. Then we have $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ and for any subdomain $\mathcal{O} \subset\subset \Omega$,

$$\|u\|_{W^{2,2}(\mathcal{O})} \le C\left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}\right)$$

for $C = C(n, \lambda, K, d)$, where λ is the constant of uniform ellipticity, $d = \mathrm{dist}(\mathcal{O}, \partial\Omega)$ and

$$K = \max\left\{\|a_{ij}\|_{C^{0,1}\left(\bar{\Omega}\right)}, \|b_j\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}\right\}.$$

◀ Since u is a weak solution,

$$\int_\Omega a_{ij}D_juD_iv = \int_\Omega gv \text{ for all } v \in W^{1,2}_0(\Omega),$$

where $g := b_jD_ju + cu - f$.

Now fix $\varphi \in C^1_c(\Omega)$ such that $0 \le \varphi \le 1$, and take

$$v := \Delta^{-h}\left(\varphi\Delta^h u\right),$$

where $\Delta^h = \Delta^h_k$ for some $1 \le k \le n$ (see Definition 6.10), where $|2h| < \mathrm{dist}(\mathrm{supp}(\varphi), \partial\Omega)$.

Observe that

$$\begin{aligned} D_iv &= \Delta^{-h}D_i\left(\varphi^2\Delta^h u\right) = \\ &= \Delta^{-h}\left(2\varphi D_i\varphi\Delta^h u + \varphi^2\Delta^h(D_iu)\right) \\ &= 2\Delta^{-h}(\varphi D_i\varphi\Delta^h u) + \Delta^{-h}(\varphi^2\Delta^h(D_iu)). \end{aligned}$$

Thus

$$\int_\Omega a_{ij}D_juD_iv = \int_\Omega 2a_{ij}D_ju\Delta^{-h}\left(\varphi D_i\varphi\Delta^h u\right) + \int_\Omega a_{ij}D_ju\Delta^{-h}\left(\varphi^2\Delta^h\left(D_iu\right)\right).$$

Let's work on the last integral: firstly by the integral identity (13),

$$\int_\Omega a_{ij}D_ju\Delta^{-h}\left(\varphi^2\Delta^h\left(D_iu\right)\right) = \int_\Omega \Delta^h\left(a_{ij}D_ju\right)\varphi^2\Delta^h\left(D_iu\right).$$

Next, since for any f_1, f_2 we have

$$\Delta^h_k(f_1f_2)(x) = f_1(x + he_k)\Delta^h_k f_2(x) + f_2(x)\Delta^h_k f_1(x),$$

in particular

$$\Delta^h\left(a_{ij}D_ju\right)(x) = a_{ij}(x + he_k)\Delta^h\left(D_ju(x)\right) + \Delta^h\left(a_{ij}(x)\right)D_ju(x).$$

Thus

$$\int_\Omega \Delta^h\left(a_{ij}D_ju\right)\varphi^2\Delta^h\left(D_iu\right) = \int_\Omega \varphi^2a_{ij}(x + he_k)\Delta^h\left(D_ju\right)\Delta^h\left(D_iu\right) + \int_\Omega \varphi^2\Delta^h(a_{ij})D_ju\Delta^h\left(D_iu\right).$$

Denote by:

$$A := \int_\Omega \varphi^2 \Delta^h(a_{ij}) D_j u \Delta^h (D_i u),$$
$$B := \int_\Omega 2a_{ij} D_j u \Delta^{-h} \left(\varphi D_i \varphi \Delta^h u\right),$$
$$C := \int_\Omega g \Delta^{-h} \left(\varphi^2 \Delta^h u\right).$$

We therefore have

$$\int_\Omega \varphi^2 a_{ij}(x+he_k) \Delta^h (D_j(u)) \Delta^h (D_i u) = C - A - B.$$

Now consider the left hand side: by the assumption on uniform ellipticity we have

$$\lambda \int_\Omega \varphi^2 |\Delta^h Du|^2 \leq \int_\Omega \varphi^2 a_{ij}(x+he_k) \Delta^h (D_j(u)) \Delta^h (D_i u).$$

Next we work on the three terms A, B and C individually. First A :

$$\begin{aligned}
|A| &= \left| \int_\Omega \varphi^2 \left(\Delta^h(a_{ij}) D_j u \Delta^h (D_i u\right) \right| \\
&\leq \epsilon \int_\Omega \varphi^2 |\Delta^h Du|^2 + \frac{1}{4\epsilon} \int_\Omega \varphi^2 |\Delta^h(a_{ij})|^2 |D_j u|^2 \quad \text{using } ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2, \\
&\leq \epsilon \int_\Omega \varphi^2 |\Delta^h Du|^2 + \frac{Cn^2K^2}{4\epsilon} \|Du\|_{L^2(\Omega)}.
\end{aligned}$$

Now B:

$$\begin{aligned}
|B| &= \left| \int_\Omega 2a_{ij} D_j u \Delta^{-h} \left(\varphi D_i \varphi \Delta^h u\right) \right| \\
&\leq 2n^2 K \|Du\|_{L^2(\Omega)} \left(\int_\Omega |\Delta^{-h} \left(\varphi D_i \varphi \Delta^h u\right) \right).
\end{aligned}$$

Now we apply the first difference quotient lemma (Proposition 6.11), noting that this is applicable as we may as well just integrate over the compact subdomain $\operatorname{supp}(\varphi)$ to obtain

$$|B| \leq 2n^2 K \|Du\|_{L^2(\Omega)} \left\| D\left(\varphi D_i \varphi \Delta^h u\right) \right\|_{L^2(\Omega)}.$$

But then

$$\begin{aligned}
\int_\Omega |D(\varphi D_i \varphi \Delta^h u)|^2 &= \int_\Omega |D(\varphi D_i \varphi) \Delta^h u + \varphi D_i \varphi \Delta^h (Du)|^2 \\
&\leq \left\| D(\varphi D_i \varphi) \Delta^h u \right\|_{L^2(\Omega)} + \sup_\Omega |D\varphi| \int_\Omega \varphi^2 |\Delta^h Du|^2 \\
&\leq \left(\sup_\Omega |D\varphi|^2 + \sup_\Omega |D^2 \varphi| \right) \left(\|Du\|_{L^2(\Omega)} + \sup_\Omega |D\varphi| \int_\Omega \varphi^2 |\Delta^h Du|^2 \right)
\end{aligned}$$

using the first difference quotient lemma several times.

Finally work on C :

$$\begin{aligned}
|C| &= \left| \int_\Omega g \Delta^{-h} (\varphi^2 \Delta^h u) \right| \leq \\
&\leq \|g\|_{L^2(\Omega)} \left\| \Delta^{-h} \left(\varphi^2 \Delta^h u\right) \right\|_{L^2(\Omega)} \\
&\leq c \left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} \right) \left\| D\left(\varphi^2 \Delta^h u\right) \right\|_{L^2(\Omega)} \\
&\leq c \left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} \right) \left(2 \sup_\Omega |D\varphi| \|Du\|_{L^2(\Omega)} + \int_\Omega \varphi^2 \left|\Delta^h Du\right|^2 \right),
\end{aligned}$$

where we used the first difference quotient lemma.

Putting this altogether we obtain

$$\begin{aligned}
\lambda \int_\Omega \varphi^2 |\Delta^h Du|^2 &\leq |C| + |A| + |B| \\
&\leq \epsilon \int_\Omega \varphi^2 |\Delta^h Du|^2 + \frac{Cn^2K^2}{4\epsilon} \|Du\|_{L^2(\Omega)} \\
&\quad + \left(\sup_\Omega |D\varphi|^2 + \sup_\Omega |D^2\varphi| \right) \left(\|Du\|_{L^2(\Omega)} + \sup_\Omega |D\varphi| \int_\Omega \varphi^2 |\Delta^h Du|^2 \right) \\
&\quad + c \left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} \right) \left(2 \sup_\Omega |D\varphi| \, \|Du\|_{L^2(\Omega)} + \int_\Omega \varphi^2 \left| \Delta^h Du \right|^2 \right).
\end{aligned}$$

Now let $\mathcal{O} \subset\subset \Omega$ be arbitrary, and choose $\varphi \in C_c^1(\Omega)$ such that $\varphi = 1$ on $\mathcal{O}$ and $|D\varphi| \leq 2/d$ and $\left|D^2\varphi\right| \leq 4/d^2$ on Ω. We then obtain an expression of the form

$$\left\|\Delta^h Du\right\|_{L^2(\Omega')} \leq C(n, \lambda, K, d) \left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} \right),$$

and we can then apply the second difference quotient lemma (Proposition 6.13) to conclude that $u \in W^{2,2}(\Omega)$ and

$$\|Du\|_{L^2(\Omega')} \leq C(n, \lambda, K, d) \left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} \right).$$

This completes the proof. ▶

8.2 Writing the equation in general form

In the situation above, the assertion $Lu = f$ is equivalent to

$$\int_\Omega a_{ij} D_j u D_i v - b_j \left(D_j u \right) v - cuv = \int_\Omega fv \text{ for all } v \in W_0^{1,2}(\Omega).$$

But now since we know that u is actually in $W_{\text{loc}}^{2,2}(\Omega)$ by Theorem 8.1, we can integrate by parts (valid as $v \in W_0^{1,2}(\Omega)$) to obtain

$$\int_\Omega \left(a_{ij} D_{ij} u - \left(b_j D_i a_{ij} \right) D_j u + cu - f \right) v = 0 \text{ for all } v \in W_0^{1,2}(\Omega),$$

and hence that

$$a_{ij} D_{ij} u - \left(b_j - D_i a_{ij} \right) D_j u + cu - f = 0 \text{ almost everywhere.}$$

This will be useful when proving global regularity later in this chapter (specifically, Step 2 in the proof of Theorem 8.7).

We note now a fact that we will use without comment many times throughout this lecture: if $\Omega \subseteq \mathbb{R}^n$ is a bounded domain and $\mathcal{O} \subset\subset \Omega$ is a subdomain then we can always find another subdomain $\mathcal{U}$ such that $\mathcal{O} \subset\subset \mathcal{U} \subset\subset \Omega$.

8.3 Addendum

We can actually improve the estimate of Theorem 8.1 and deduce that we can bound the $W^{2,2}(\mathcal{O})$ norm of u with the $L^2(\Omega)$ norm, not the $W^{1,2}(\Omega)$ norm. More precisely, under the same conditions of Theorem 8.1, we have $u \in W_{\text{loc}}^{2,2}(\Omega)$ and for any $\mathcal{O} \subset\subset \Omega$ we have

$$\|u\|_{W^{2,2}(\mathcal{O})} \leq C \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right) \text{ for } C = C(n, \lambda, K, d).$$

◀ Indeed, given $\mathcal{O} \subset\subset \Omega$, choose $\mathcal{U}$ such that $\mathcal{O} \subset\subset \mathcal{U} \subset\subset \Omega$. Then the argument of Theorem 8.1 shows that

$$\|u\|_{W^{2,2}(\mathcal{O})} \leq C \left(\|u\|_{W^{1,2}(\mathcal{U})} + \|f\|_{L^2(\mathcal{U})} \right)$$

for an appropiate constant C. Choose a new cutoff function $\varphi \in C_c^\infty(\Omega)$ such that $\varphi = 1$ on $\mathcal{U}$ and $0 \leq \varphi \leq 1$. Now consider $v = \varphi^2 u \in W_0^{1,2}(\Omega)$: we have

$$\int_\Omega a_{ij} D_j u D_i v = \int_\Omega g v,$$

where g is as in the proof of Theorem 7.8, and thus

$$\lambda \int_\Omega \varphi^2 |Du|^2 \leq C \int_\Omega \varphi^2 \left(|f| + |Du| + |u| \right) |u|,$$

and then by repeated use of Hölder's inequality gives an expression of the form

$$\int_\Omega \varphi^2 |Du|^2 \leq C \int_\Omega f^2 + u^2.$$

Thus we obtain

$$\|u\|_{W^{1,2}(\mathcal{U})} \leq C \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right),$$

and thus

$$\|u\|_{W^{2,2}(\mathcal{O})} \leq C \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right)$$

as required. ▶

After $W^{2,2}$-regularity the next natural step is to try for $W^{3,2}$-regularity. In fact, as is often the case, once $W^{2,2}$-regularity is established, it is often much easier to obtain higher regularity. Clearly the solution cannot possess more regularity than the coefficients of L allow; we shall now see below that this is only obstruction.

8.4 Theorem (higher interior regularity)

Let $u \in W^{1,2}(\Omega)$ be a weak solution of the equation $Lu = f$, where L is uniformly elliptic in the bounded domain $\Omega \subseteq \mathbb{R}^n$,

$$Lu = D_i(a_{ij} D_j u) + b_j D_j u + cu,$$

where the $a_{ij} \in C^{k,1}\left(\bar{\Omega}\right)$ and the $b_j, c \in C^{k-1,1}\left(\bar{\Omega}\right)$ and $f \in W^{k,2}(\Omega)$. Then we have $u \in W_{\text{loc}}^{k+2,2}(\Omega)$ and for any subdomain $\mathcal{O} \subset\subset \Omega$,

$$\|u\|_{W^{k+2,2}(\mathcal{O})} \leq C \left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{W^{k,2}(\Omega)} \right)$$

for $C = C(n, \lambda, K, d)$, where λ is the constant of uniform ellipticity, $d = \operatorname{dist}(\mathcal{O}, \partial\Omega)$ and

$$K = \max \left\{ \|a_{ij}\|_{C^{k,,1}(\bar{\Omega})}, \|b_j\|_{C^{k-1,1}(\bar{\Omega})}, \|c\|_{C^{k-1,1}(\bar{\Omega})} \right\}.$$

◀ Induction on k. The case $k = 0$ is precisely Theorem 8.1. For notational simplicity will shall prove the case $k = 1$ given the case $k = 0$; the general inductive step is similar (only we integrate by parts many times and the function $\hat{g}$ is somewhat messier).

Thus by what already have $u \in W_{\text{loc}}^{2,2}(\Omega)$, and if $\mathcal{O} \subset\subset \Omega$ then there exists C depending on the appropiate things such that

$$\|u\|_{W^{2,2}(\mathcal{O})} \leq C \left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{W^{0,2}(\Omega)} \right).$$

Take $\mathcal{U} \subset\subset \mathcal{O}$ and let $1 \leq \ell \leq n$. Choose any test function $v \in C_c^\infty(\mathcal{O})$, and set $w = D_\ell \varphi$. Then we have

$$\int_\Omega a_{ij} D_j u D_i w = \int_\Omega g w.$$

Now we integrate by parts to obtain

$$\int_\Omega a_{ij} D_j(D_\ell u) D_i v = \int_\Omega \hat{g} v,$$

where

$$-D_\ell a_{ij} D_j u + D_\ell g.$$

By approximation this actually holds for all $v \in W_0^{1,2}(\mathcal{O})$.

Next, we claim that $\hat{g} \in L^2(O)$ - this follows immediately given the hypotheses on the coefficients and the fact that $u \in W^{2,2}(\mathcal{O})$. Thus we can apply Theorem 8.1, noting that $\|\hat{g}\|_{L^2(\mathcal{O})} \leq C\left(\|u\|_{W^{2,2}(\mathcal{O})} + \|f\|_{W^{1,2}(\Omega)}\right)$, to conclude that $D_\ell u \in W^{2,2}(\mathcal{U})$ and

$$\|D_\ell u\|_{W^{2,2}(\mathcal{U})} \leq C\left(\|u\|_{W^{2,2}(\mathcal{O})} + \|f\|_{L^2(\mathcal{O})}\right).$$

Since ℓ was arbitrary, we conclude that $u \in W^{3,2}_{\mathrm{loc}}(\Omega)$, and that we have an estimate of the desired form

$$\|u\|_{W^{3,2}(\mathcal{U})} \leq C\left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{W^{1,2}(\Omega)}\right).$$

This completes the inductive step (with $k = 1$) and thus the proof. ▶

8.5 Corollary (smoothness in the interior)

Suppose that $u \in W^{1,2}(\Omega)$ is a weak solution of the equation $Lu = f$, where L is uniformly elliptic in the bounded domain $\Omega \subseteq \mathbb{R}^n$,

$$Lu = D_i(a_{ij} D_j u) + b_j D_j u + cu,$$

where the $a_{ij}, b_j, c, f \in C^\infty\left(\bar{\Omega}\right)$. Then $u \in C^\infty(\Omega)$, that is, u is a classical solution to $Lu = f$ in Ω.

◀ By Theorem 8.4, $u \in W^{k,2}_{\mathrm{loc}}(\Omega)$ for all $k \in \mathbb{N}$. Hence by the general Sobolev inequality Theorem 6.14.2 we have $u \in C^m(\Omega)$ for each $m \in \mathbb{N}$. Thus $u \in C^\infty(\Omega)$. ▶

We now wish to discuss the regularity of weak solutions on the boundary, which will allows us to prove global regularity of weak solutions. First however we mention a trick we will need in the proof of Theorem 8.7 below.

8.6 Straightening out the boundary

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain such that $\partial\Omega$ is C^2. Fix $x_0 \in \partial\Omega$. Then (possibly after relabelling and reorienting the coordinate axes) there exists $r > 0$ and a C^2 function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$B_r(x_0) \cap \partial\Omega = \{x \mid \gamma(x') < x_n\},$$

where we use the notation $x' := (x_1, \ldots, x_{n-1})$. Consider now the functions $\psi, \eta : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\psi(x) = (x', x_n - \gamma(x')),$$

$$\eta(y) = (y', y_n + \gamma(y)).$$

Then $\eta = \psi^{-1}$ and ψ 'straightens out' the boundary of $\partial\Omega$ near x_0, mapping $\partial\Omega$ onto an upper half plane. Note that the Jacobian of ψ, $J(\psi)$ is identically 1.

8.7 Theorem (global $W^{2,2}(\Omega)$ regularity)

Let $u \in W^{1,2}(\Omega)$ be a weak solution of the equation $Lu = f$, where L is uniformly elliptic in the bounded domain $\Omega \subseteq \mathbb{R}^n$,

$$Lu = D_i(a_{ij}D_ju) + b_iD_iu + cu,$$

where the $a_{ij} \in C^{0,1}\left(\bar{\Omega}\right)$, the b_j and c are in $L^\infty(\Omega)$ and $f \in L^2(\Omega)$.

Moreover, assume that $\partial\Omega$ is of class C^2 and that there exists a function $\varphi \in W^{2,2}(\Omega)$ for which $u - \varphi \in W_0^{1,2}(\Omega)$.

Then we have $u \in W^{2,2}(\Omega)$ and,

$$\|u\|_{W^{2,2}(\Omega)} \leq C\left(\|u\|_{L^2\Omega)} + \|f\|_{L^2(\Omega)} + \|\varphi\|_{W^{2,2}(\Omega)}\right)$$

for $C = C(n, \lambda, K, \partial\Omega)$, where λ is the constant of uniform ellipticity and

$$K = \max\left\{\|a_{ij}\|_{C^{0,1}(\overline{\Omega})}, \|b_j\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}\right\}.$$

◄ We will prove this in two steps.

Step 1: Reducing to a special case.

Replacing u by $u - \varphi$ we see as in Step 1 of the proof of Theorem 7.8 that with no loss of generality we may assume $\varphi = 0$ and hence $u \in W_0^{1,2}(\Omega)$. Then by the argument in Addendum 8.3 we may estimate

$$\|u\|_{W^{1,2}(\Omega)} \leq C\left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right),$$

where $C = C(n, \lambda, K)$.

Since $\partial\Omega$ is C^2, using the trick in Section 8.6 above, there exists for each point $x_0 \in \partial\Omega$ a ball $B = B_r(x_0)$ and a bijective mapping $\psi \in C^2(B)$ from B onto an open set $D \subseteq \mathbb{R}^n$ such that

$$\psi(B \cap \Omega) \subseteq \mathbb{R}^n_+ =: \{y \in \mathbb{R}^n \mid y_n > 0\},$$

and $\psi(B \cap \partial\Omega) = \partial\mathbb{R}^n_+$, with $\psi^{-1} \in C^2(D)$. Let $B_\rho(x_0) \subset\subset B$, and set

$$B^+ := B_\rho(x_0) \cap \Omega,$$

$$D^{'} := \psi(B_\rho(x_0)),$$

and $D^+ := \psi(B^+)$.

Now define $\tilde{u}(y) := u(x)$, where $\psi(x) = y$. Define $\tilde{L}$ by $\tilde{L}\tilde{u}(y) = Lu(x)$. Explicitly,

$$\tilde{L}\tilde{u} = D_i\left(\tilde{a}_{ij}D_j\tilde{u}\right) + \tilde{b}_iD_i\tilde{u} + \tilde{c}\tilde{u} = \tilde{f}(y),$$

where

$$\tilde{a}_{ij}(y) = \frac{\partial\psi_i}{\partial\psi_r}\frac{\partial\psi_i}{\partial\psi_s}a_{ij}(x), \quad \tilde{b}_i(y) = \frac{\partial^2\psi_i}{\partial x_r\partial x_s}a_{rs}(x) + \frac{\partial\psi_i}{\partial x_r}b_r(x),$$

and

$$\tilde{c}(y) = c(x), \quad \tilde{f}(y) = f(x).$$

We certainly have $\tilde{b}_j, \tilde{c} \in L^\infty(\psi(B \cap \Omega))$ and $\tilde{f} \in L^2(\psi(B \cap \Omega))$. Moreover it is clear that $\tilde{a}_{ij} \in C^{0,1}\left(\overline{\psi(B \cap \Omega)}\right)$. Let us check that $\tilde{L}$ is still uniformly elliptic in $\psi(B \cap \Omega)$. Indeed, given any $\xi \in \mathbb{R}^n$,

$$\begin{aligned}
\sum_{i,j=1}^n \tilde{a}_{ij}(y)\xi_i\xi_j &= \sum_{i,j,r,s=1}^n a_{rs}(x)\frac{\partial\psi_i}{\partial\psi_r}\frac{\partial\psi_j}{\partial\psi_s}\xi_i\xi_j \\
&= \sum_{r,s=1}^n a_{rs}(x)\frac{\partial(\psi\cdot\xi)}{\partial\psi_r}\frac{\partial(\psi\cdot\xi)}{\partial\psi_s} \\
&\geq \lambda|D\psi\cdot\xi|^2 \\
&\geq \tilde{\lambda}\,\|\xi\|,
\end{aligned}$$

where to see that last inequality write $\bar{\xi} := D\psi \cdot \xi$, and so $\xi = D\varphi \cdot \bar{\xi}$, and thus $\|\xi\| \leq C \left\|\bar{\xi}\right\|$ for some $C = C(\psi, \gamma)$.

Next note that since $u \in W_0^{1,2}(\Omega)$, the transformed solution $v = u \circ \psi^{-1}$ is in $W^{1,2}(D^+)$ and satisfies $\varphi v \in W_0^{1,2}(D^+)$ for all $\varphi \in C_0^1(D')$.

Step 2: Applying the previous results.

We will now just write u instead of $\tilde{u}$ etc. in order to simplify the notation. Accordingly, let us suppose that $u \in W^{1,2}(D^+)$ satisifies satisfies $Lu = f$ in D^+ and that $\varphi u \in W_0^{1,2}(D^+)$ for any $\varphi \in C_0^1(D')$. Then for $|h| < \operatorname{dist}(\operatorname{supp}(\varphi), \partial D')$ and $1 \leq k \leq n-1$, we have $\varphi^2 \Delta_k^h u \in W_0^{1,2}(D^+)$. Consequently the proof of Theorem 8.1 will apply and we may conclude that $D_{ij}u \in L^2(\psi(B_\rho(x_0) \cap \Omega))$ as long as $i + j < 2n$.

The remaining deriviative $D_{nn}u$ can be estimated directly using the ideas of Section 8.2: explicitly since we have

$$a_{nn}D_{nn}u + (b_m - D_n a_{nn})D_n u + cu - f = 0,$$

almost everywhere we have

$$D_{nn}u = -\frac{1}{\lambda}\left((b_n - D_n a_{nn})\, D_n u + cu - f - \sum_{i+j<2n} D_{ij}u\right),$$

and hence we have (where all norms are taken with respect to $L^2(\psi(B_\rho(x_0) \cap \Omega))$),

$$\|D_{nn}u\| \leq C\left(\sum_{i,j<2n} \|D_{ij}u\| + \|Du\| + \|u\| + \|f\|\right).$$

Thus we have bounds on all the derivatives.

Step 3: Returning to Ω.

Hence, returning to the original domain Ω with the mapping $\psi^{-1} \in C^2(D)$ we obtain that $u \in W^{2,2}(B_\rho(x_0) \cap \Omega)$. Since x_0 was an arbitrary point of $\partial\Omega$ and by Theorem 8.1 we already have $u \in W^{2,2}_{\text{loc}}(\Omega)$ we infer that $u \in W^{2,2}(\Omega)$. Finally, by choosing a finite number of points x_i in the compact set $\partial\Omega$ such that the balls $B_\rho(x_i)$ cover $\partial\Omega$, we obtain the estimate of the theorem from Theorem 8.1 and the estimate in Step 1. The proof is complete. ►

As before, once we have established $W^{2,2}$-regularity it is comparitively easy to prove as much regularity as the coefficients of L allow.

8.8 Theorem (higher global regularity)

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with C^{k+2} boundary. Let $u \in W^{1,2}(\Omega)$ be a weak solution of the equation $Lu = f$, $u - \varphi \in W_0^{1,2}(\Omega)$, where L is uniformly elliptic in Ω,

$$Lu = D_i(a_{ij}D_j u) + b_j D_j u + cu,$$

and $\varphi \in W^{k+2,2}(\Omega)$, $a_{ij} \in C^{k,1}\left(\bar{\Omega}\right)$, $b_j, c \in C^{k-1,1}\left(\bar{\Omega}\right)$ and $f \in W^{k,2}(\Omega)$. Then we have $u \in W^{k+2,2}(\Omega)$ and

$$\|u\|_{W^{k+2,,2}(\Omega)} \leq C\left(\|u\|_{L^2(\Omega)} + \|f\|_{W^{k,2}(\Omega)} + \|\varphi\|_{W^{k+2,2}(\Omega)}\right)$$

for $C = C(n, \lambda, K, k, \partial\Omega)$, where λ is the constant of uniform ellipticity and

$$K = \max\left\{\|a_{ij}\|_{C^{k,,1}(\bar{\Omega})}, \|b_j\|_{C^{k-1,1}(\bar{\Omega})}, \|c\|_{C^{k-1,1}(\bar{\Omega})}\right\}.$$

◄ Induction on k; the case $k = 0$ is Theorem 8.7. Observe that, using the notation from the proof of Theorem 8.7, the conditions $u \in W^{2,2}(D^+)$, $\eta u \in W_0^{1,2}(\Omega)$ for all $\eta \in C_c^1(D')$ imply that

$\eta D_k u \in W_0^{1,2}(\Omega)$ for $1 \leq k \leq n-1$. Indeed, the first difference quotient lemma (Proposition 6.11) gives us $\eta\Delta_k^h u \in W_0^{1,2}(D^+)$ and

$$\left\|\eta\Delta_k^h u\right\|_{W^{1,2}(D^+)} \leq \|\eta\|_{C^1(D^+)} \|u\|_{W^{2,2}(D^+)}$$

for sufficiently small h. Thus choosing $h_m \downarrow 0$ we obtain a subsequence $\{h_{m'}\}$ such that $\left\{\eta\Delta_k^{h_{m'}} u\right\}$ is weakly convergent in the Hilbert space $W_0^{1,2}(\Omega)$. The limit of this sequence is clearly the function $\eta D_k u$. Further global regularity then follows as in Theorem 8.4. ▶

8.9 Corollary (global smoothness)

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ of class C^∞. Suppose that $u \in W^{1,2}(\Omega)$ be a weak solution of the equation $Lu = f$, $u - \varphi \in W_0^{1,2}(\Omega)$ where L is uniformly elliptic in Ω

$$Lu = D_i(a_{ij}D_j u) + b_j D_j u + cu,$$

and $a_{ij}, b_j, c, f, \varphi \in C^\infty\left(\bar{\Omega}\right)$. Then $u \in C^\infty\left(\bar{\Omega}\right)$.

◀ Identical to the proof of Corollary 8.5. ▶

9 The Euler-Lagrange equations

In this final chapter we return to the opening chapter's brief description of the Euler-Lagrange equations and variational problems (see Sections 1.4 and 1.5). We conclude the course by briefly outlining the method of **elliptic bootstrapping**.

9.1 The direct method of calculus of variations

Let us return to the Dirichlet problem: $\Delta u = 0$ in Ω, with $u - \varphi \in W_0^{1,2}(\Omega)$, for some $\varphi \in W^{1,2}(\Omega)$. Set

$$\mathcal{F}(v) := \int_\Omega |Dv|^2,$$

and let $\mathcal{C} := \{v \in W^{1,2}(\Omega) \mid v - \varphi \in W_0^{1,2}(\Omega)\}$. Note that $\mathcal{C} \neq \emptyset$, as $\varphi \in \mathcal{C}$. We wish to minimise $\mathcal{F}$ in the class $\mathcal{C}$.

Set

$$m := \inf_{v\in\mathcal{C}} \mathcal{F}(v),$$

and take a minimising sequence $\{v_k\}$. Note that $m \geq 0$ (and so in particular, $m > -\infty$).

We may assume there exists $C \geq 0$ such that $\mathcal{F}(v_k) \leq C$ for all k. Hence

$$\int_\Omega |D(v_k - \varphi)|^2 \leq C + \|D\varphi\|_{L^2(\Omega)},$$

and thus by Poincaré's inequality (Proposition 5.20), since $v_k - \varphi \in W_0^{1,2}(\Omega)$ we have

$$\|v_k - \varphi\|_{L^2(\Omega)} \leq C\,\|D(v_k - \varphi)\|_{L^2(\Omega)},$$

and thus $\{v_k\}$ is a bounded sequence in the Hilbert space $W^{1,2}(\Omega)$. Thus there exists a weakly convergent subsequence $v_k' \rightharpoonup u \in W^{1,2}(\Omega)$ (see Section 6.12). Then $v_k \rightharpoonup u$ and $Dv_{k'} \rightharpoonup Du$ in $L^2(\Omega)$, and since a closed convex subset of a normed space is weakly closed, the norm is weakly semicontinuous, and we conclude that

$$\int_\Omega |Du|^2 \leq \liminf_{k'\to\infty} \int_\Omega |Dv_{k'}|^2 = m,$$

and hence u does indeed minimise $\mathcal{F}$.

By taking the first variation, we see that

$$0 = \frac{d}{ds}\left(\mathcal{F}(u+s\eta)\right)\Big|_{s=0} = \int_\Omega Du\cdot D\eta \text{ for all } C^\infty(\Omega),$$

and hence $u \in C^\infty(\Omega)$ by Theorem 2.12. Thus we have (once again) solved the Dirichlet problem for the Laplacian, this time by what is called the **direct method of calculus of variations**. We now wish to study more general variational problems.

9.2 Definition

Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be smooth, where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary. Let $1 < q < \infty$.

We say that F is a **Lagrangian** if F satisfies the following two conditions:

1. **Coercivity**: there exists constants $\alpha > 0, \beta \in \mathbb{R}$ such that

$$F(x,z,p) \geq \alpha|p|^q - \beta \text{ for all } x \in \Omega, z \in \mathbb{R} \text{ and } p \in \mathbb{R}^n;$$

   in particular F is bounded below.

2. **Convexity in the 'p' variable**: the mapping

$$p \mapsto F(x,z,p)$$

   is convex for each $x \in \Omega, z \in \mathbb{R}$. That is, for each $\xi \in \mathbb{R}^n$,

$$\sum_{i,j=1}^n F_{p_ip_j}(x,z,p)\xi_i\xi_j \geq 0.$$

   Equivalently, the Hessian matrix $\left[F_{p_ip_j}(x,z,p)\right]$ should be positive semi-definite for each $x \in \Omega$ and $z \in \mathbb{R}$.

We then study as before the variational functional (♠)

$$\mathcal{F}(u) := \int_\Omega F(x,u,Du)dx$$

for $u : \Omega \to \mathbb{R}$, and look for minimizers of $\mathcal{F}$ within a given class.

Note that coercivity implies that:

$$\mathcal{F}(v) \geq \alpha\,\|Dv\|^q_{L^q(\Omega)} - \beta|\Omega|. \tag{14}$$

Hence $\mathcal{F}(v) \to \infty$ as $\|Dv\|_{L^q(\Omega)} \to \infty$. The next section explains why the convexity assumption is a natural one to make.

9.3 The second variation

Suppose we know that there exists a smooth minimizer u of

$$\mathcal{F}(v) = \int_\Omega F(x,u,Du).$$

Then let $\varphi \in C_c^\infty(\Omega)$ be arbitrary and set

$$i(s) := \mathcal{F}(u+s\varphi).$$

Then we have

$$i'(0) = 0, i''(0) \geq 0.$$

Let us compute the **second variation** $i''(s)$:

$$i''(s) = \int_\Omega F_{p_ip_j}(x,u+s\varphi,Du+sD\varphi)D_i\varphi D_j\varphi + 2F_{p_iz}(x,u+s\varphi,Du+sD\varphi)\varphi D_i\varphi + F_{zz}(x,u+s\varphi,Du+sD\varphi)\varphi^2,$$

and thus setting $s=0$ gives

$$0 \leq i''(0) = \int_\Omega F_{p_ip_j}(x,u,Du)D_i\varphi D_j\varphi + F_{p_iz}(x,u,Du)\varphi D_i\varphi + F_{zz}(x,u,D\varphi)\varphi^2. \tag{15}$$

By approximation this actually holds for any Lipschitz continuous function v vanishing on $\partial\Omega$. Fix an arbitrary $\xi \in \mathbb{R}^n$ and set

$$\rho(x) := \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ 1-x & \frac{1}{2} \leq x \leq 1 \end{cases} \quad \text{and } \rho(x+1) = \rho(x) \text{ for all } x \in \mathbb{R}.$$

Note that $|\rho'| = 1$ almost everywhere. Now set

$$v(x) := \epsilon\rho\left(\frac{x\cdot\xi}{\epsilon}\right)\varphi(x),$$

where $\varphi \in C_c^\infty(\Omega)$. Then v is a Lipschitz continuous function vanishing on $\partial\Omega$, and note that

$$\begin{aligned} D_iv &= \rho'\left(\frac{x\cdot\xi}{\epsilon}\right)\xi_i\varphi(x) + \epsilon\rho\left(\frac{x\cdot\xi}{\epsilon}\right)D_i\varphi \\ &= \rho'\left(\frac{x\cdot\xi}{\epsilon}\right)\xi_i\varphi(x) + O(\epsilon) \text{ as } \epsilon \to 0. \end{aligned}$$

Thus substituting v into equation (15) gives

$$0 \leq \int_\Omega F_{p_ip_j}(x,u,Du)(\rho')^2\xi_i\xi_j\varphi^2 + O(\epsilon).$$

Since $|\rho'| = 1$ almost everywhere, letting $\epsilon \to 0$ gives us

$$0 \leq \int_\Omega F_{p_ip_j}(x,u,Du)\xi_i\xi_j\varphi^2,$$

and then since this holds for all $\varphi \in C_c^\infty(\Omega)$, we deduce that

$$\sum_{i,j=1}^n F_{p_ip_j}(x,u,Du)\xi_i\xi_j \geq 0.$$

Thus the convexity assumption above is a necessary one in order to allow for the existence of a smooth minimizer.

We now state and prove two theorems which allow us to use the direct method of the calculus of variations to solve the variational problem (♠) for a Lagrangian.

9.4 Theorem (weak lower semicontinuity)

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a Lagrangian. Then $\mathcal{F}(\cdot)$ is weakly lower semicontinuous in $W^{1,q}(\Omega)$.

◀ We will actually prove this under the weaker assumption that if F is bounded below and convex in the 'p' variable then $\mathcal{F}(\cdot)$ is weakly lower semicontinuous; this result does not need the full strength of the coercivity condition from Definition 9.2.

Suppose $\{u_k\} \subseteq W^{1,2}(\Omega)$ is such that u_k converges weakly to u in $W^{1,q}(\Omega)$. Set

$$m := \liminf_{k\to\infty} \mathcal{F}(u_k).$$

We must show that $\mathcal{F}(u) \leq m$. By definition of $W^{1,q}(\Omega)$ weak convergence we have $u_k \rightharpoonup u$ and $Du_k \rightharpoonup u_k$ weakly in $L^q(\Omega)$. Since the sequence $\{u_k\}$ is weakly convergent in $W^{1,q}(\Omega)$ the u_k are uniformly bounded in $W^{1,q}(\Omega)$, that is

$$\sup_k \|u_k\|_{W^{1,q}(\Omega)} < \infty.$$

Thus passing to a subsequence if necessary, we may assume that

$$m = \lim_{k\to\infty} \mathcal{F}(u_k).$$

Furthermore, by Theorem 6.16 we have $W^{1,q}(\Omega) \subset\subset L^q(\Omega)$, and since $\{u_k\}$ is a uniformly bounded sequence, we have $u_k \to u$ strongly in Ω, and hence by passing to another subsequence if necessary, we may assume that $u_k(x) \to u(x)$ for a.e. $x \in \Omega$.

Now fix $\epsilon > 0$. Egoroff's theorem (see Section 1.3) asserts that there exists a measurable set E_ϵ such that $|\Omega \backslash E_\epsilon| \leq \epsilon$ and $u_k \to u$ uniformly on E_ϵ. Now set

$$F_\epsilon := \left\{ x \in \Omega \mid |u(x)| + |Du(x)| \leq \frac{1}{\epsilon} \right\}.$$

Then $|\Omega\backslash F_\epsilon| \to 0$ as $\epsilon \to 0$, and hence if we set

$$G_\epsilon := E_\epsilon \cap F_\epsilon$$

then $|\Omega \backslash G_\epsilon| \to 0$ as $\epsilon \to 0$.

Now assume that since F is bounded below, by adding a constant if necessary we may as well assume that $F \geq 0$. Now by convexity,

$$F(x, u_k, Du_k) \leq F(x, u_k, Du) + D_p F(x, u_k, Du) \cdot (Du_k - Du)$$

for each $x \in \Omega$, and thus integrating gives

$$\begin{aligned} \mathcal{F}(u_k) &= \int_\Omega F(x, u_k, Du_k) \\ &\geq \int_{G\epsilon} F(x, u_k, Du_k) \\ &\geq \int_{G_\epsilon} F(x, u_k, Du) + \int_{G_\epsilon} F(x, u_k, Du) \cdot (Du_k - Du). \end{aligned}$$

Since $u_k \to u$ uniformly and $|u(x)| + |Du(x)| \leq \frac{1}{\epsilon}$ on G_ϵ, we may apply the dominated convergence theorem to conclude that

$$\lim_{k\to 0} \int_{G_\epsilon} F(x, u_k, Du) = \int_{G_\epsilon} F(x, u, Du).$$

Moreoever, since $D_p F(x, u_k, Du) \to D_p F(x, u, Du)$ uniformly on G_ϵ and $Du_k \rightharpoonup Du$ weakly in $L^q(\Omega)$, we have

$$\lim_{k\to 0} \int_{G_\epsilon} D_p F(x, u_k, Du) \cdot (Du_k - Du) = 0.$$

We thus have

$$m = \lim_{k\to 0} \mathcal{F}(u_k) \geq \int_{G_\epsilon} F(x, u, Du).$$

This holds for each $\epsilon > 0$. We now let ϵ tend to zero, and then using the fact that F is non-negative, by the monotone convergence theorem we conclude

$$m \geq \int_\Omega F(x, u, Du) = \mathcal{F}(u)$$

as required. ►

9.5 Theorem (existence of minimizer of (♠))

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a Lagrangian, and $\mathcal{F}$ the functional (♠). Suppose $\varphi \in W^{1,q}(\Omega)$ is given. Then there exists u in the set $\mathcal{C} := \{v \in W^{1,q}(\Omega) \mid v - \varphi \in W_0^{1,q}(\Omega)\}$ such that

$$\mathcal{F}(u) = \inf_{v \in \mathcal{C}} \mathcal{F}(v).$$

◀ Set $m := \inf_{v \in \mathcal{C}} \mathcal{F}(v)$. We may clearly assume that m is finite. Select a minimising sequence $\{u_k\}$. By adding a constant if necessary, we may assume that $\beta = 0$ in (14). Thus $F \geq \alpha |p|^q$ and hence

$$\mathcal{F}(w) \geq \alpha \int_\Omega |Dw|^q.$$

Since m is finite, this shows that

$$\sup_k \|Du_k\|_{L^q(\Omega)} < \infty.$$

Now fix any $w \in \mathcal{C}$. Then $u_k - w \in W_0^{1,q}(\Omega)$ and hence by Poincaré's inequality (Proposition 5.20) we have

$$\begin{aligned} \|u_k\|_{L^q(\Omega)} &\leq \|u_k - w\|_{L^q(\Omega)} + \|w\|_{L^q(\Omega)} \\ &\leq C \|Du_k - Dw\|_{L^q(\Omega)} + C', \end{aligned}$$

and thus

$$\sup_k \|u_k\|_{L^q(\Omega)} < \infty.$$

Thus the sequence $\{u_k\}$ is bounded in $W^{1,q}(\Omega)$. Passing to a subsequence if necessary, we may therefore that $\{u_k\}$ is weakly convergent to some $u \in W^{1,q}(\Omega)$. Next claim that $u \in \mathcal{C}$: since $W_0^{1,q}(\Omega)$ is a closed linear subspace of $W^{1,q}(\Omega)$, $W_0^{1,q}(\Omega)$ is weakly closed (this is **Mazur's theorem**), and hence $u - \varphi \in W_0^{1,q}(\Omega)$ since the same is true of each $u_k - \varphi$.

Then by Theorem 9.4,

$$\mathcal{F}(u) \leq \liminf_k \mathcal{F}(u_k) = m = \inf_{v \in \mathcal{C}} \mathcal{F}(v).$$

The proof is complete. ▶

The next thing to do is show that under suitable growth conditions on F, the minimizers of (♠) do indeed solve the Euler-Lagrange equations.

9.6 Definition

We say that $u \in \mathcal{C} = \{v \in W^{1,q}(\Omega) | u - \varphi \in W_0^{1,q}(\Omega)\}$ is a **weak solution** of the associated Euler-Lagrange equations (5) (see Section 1.5) if

$$\int_\Omega F_{p_i}(x, u, Du) D_i v + F_z(x, u, Du) v = 0 \quad \text{for all } v \in W_0^{1,q}(\Omega).$$

9.7 Proposition

Suppose the Lagrangian F satisfies the following growth condition: there exists $C \geq 0$ such that for all $x \in \Omega$, $z \in \mathbb{R}$ and $p \in \mathbb{R}^n$,

1. $|F(x, z, p)| \leq C \left(|z|^{q-1} + |p|^{q-1}| + 1 \right)$,
2. $|D_p F(x, z, p)| \leq C \left(|z|^{q-1} + |p|^{q-1}| + 1 \right)$,
3. $|D_z F(x, z, p)| \leq C \left(|z|^{q-1} + |p|^{q-1}| + 1 \right)$.

Then if $u \in \mathcal{C}$ is a minimizer of $\mathcal{F}(\cdot)$ then u is a weak solution to the associated Euler-Lagrange equations.

◀ Fix some $v \in W_0^{1,q}(\Omega)$. Set

$$i(s) := \mathcal{F}(u + sv).$$

Then by condition (1), $i(s)$ is finite for all $s \in \mathbb{R}$. Now take $s \neq 0$, and consider the difference quotient

$$\begin{aligned}\frac{i(s) - i(0)}{s} &= \int_\Omega \frac{1}{s}\left(F(x, u + sv, Du + sDv) - F(x, u, Du)\right) \\ &= \int_\Omega F^s(x)dx,\end{aligned}$$

where

$$F^s(x) := \frac{1}{s}\left(F(x, u + sv, Du + sDv) - F(x, u, Du)\right)$$

for a.e. $x \in \Omega$.

Clearly

$$F^s(x) \to \sum_{i=1}^n F_{p_i}(x, u, Du)D_i v + F_z(x, u, Du)v$$

almost everywhere as $s \to 0$.

Now by Young's inequality

$$ab \leq \frac{a^r}{r} + \frac{b^q}{q} \text{ for } \frac{1}{r} = 1 - \frac{1}{q},$$

and thus we can use the hypotheses (2) and (3) to conclude that

$$\begin{aligned}|F^s(x)| &\leq \frac{|D_p F(x, u + sv, Du + sDv)|^r}{r} + \frac{|Dv|^q}{q} + \frac{|D_z F(x, u + sv, Du + sDv)|^r}{r} + \frac{|v|^q}{q} \\ &\leq \frac{\left(|u + sv|^{q-1} + |Du + sDv|^{q-1}\right)^{\frac{q}{q-1}}}{\frac{q}{q-1}} + \frac{|Dv|^q}{q} + \frac{\left(|u + sv|^{q-1} + |Du + sDv|^{q-1}\right)^{\frac{q}{q-1}}}{\frac{q}{q-1}} + \frac{|v|^q}{q} \\ &= C\left(|Du|^q + |u|^q + |Dv|^q + |v|^q + 1\right).\end{aligned}$$

Since $u, v \in W^{1,q}(\Omega)$ we conclude that $C\left(|Du|^q + |u|^q + |Dv|^q + |v|^q + 1\right) \in L^q(\Omega)$ and hence we may apply the dominated convergence theorem to conclude that $i'(0)$ exists and is equal to

$$\int_\Omega F_{p_i}(x, u, Du)D_i v + F_z(x, u, Du)v.$$

But since $i(\cdot)$ has a minimum at 0 by assumption, we can thus conclude $i'(0) = 0$, and hence u is indeed a weak solution. ▶

We will now give a short survey on the regularity of minimizers.

9.8 Regularity of minimizers

Let is make the following simplifying assumptions on our Lagrangian F:

- F is a function only of p.
- F is at least C^2.
- $|D_p^2 F(p)| \leq C$ for all $p \in \mathbb{R}^n$ (this is stonger than the condition required in Proposition 9.7, that is, $|D_p F(p)| \leq C(|p| + 1)$, but is necessary for the next theorem).

- F is **uniformly convex**, tht is, there exists $\theta > 0$ such that for all $\xi \in \mathbb{R}^n$

$$\sum_{i,j=1}^{n} F_{p_i p_j}(p)\xi_i \xi_j \geq \theta |\xi^2|.$$

Then by Proposition 9.7, any minimizer $u \in \mathcal{C} := \{v \in W^{1,2}(\Omega) \mid v - \varphi \in W_0^{1,2}(\Omega)\}$, where φ is a given function in $W^{1,2}(\Omega)$ is a weak solution to the Euler-Lagrange equation

$$D_i \left(F_{p_i}(Du)\right) = 0 \text{ in } \Omega,$$

that is, for each $v \in W_0^{1,2}(\Omega)$ we have

$$\int_\Omega F_{p_i}(Du) D_i v = 0.$$

As a first step to proving additional regularity of u we have:

9.9 Theorem (interior $W^{2,2}(\Omega)$ regularity)

Assume the hypotheses of Section 9.8. Suppose $u \in W^{1,2}(\Omega)$ is a weak solution to the Euler Lagrange equation $D_i(F_{p_i}(Du)) = 0$ in Ω. Then $u \in W^{2,2}_{\text{loc}}(\Omega)$.

◀ Fix some subdomain $\mathcal{O} \subset\subset \Omega$, and then choose a subdomain $\mathcal{U}$ such that $\mathcal{O} \subset\subset \mathcal{U} \subset\subset \Omega$. Select a smooth cutoff function $\varphi \in C_c^\infty(\Omega)$ satisfying $\varphi = 1$ on $\mathcal{O}$ and $\varphi = 0$ on $\mathbb{R}^n \backslash \mathcal{U}$. Choose $|h|$ small and set

$$v := \Delta^{-h}(\varphi^2 \Delta^h u),$$

where $\Delta^h u = \Delta_k^h u$ for some $1 \leq k \leq n$. Using the integral identity (13) we deduce that

$$\int_\Omega \Delta^h \left(F_{p_i}(Du)\right) D_i(\varphi^2 \Delta^h u) = 0.$$

Now observe that

$$\begin{aligned}
\Delta^h \left(F_{p_i}(Du(x))\right) &= \frac{F_{p_i}(Du(x+he_k)) - F_{p_i}(Du(x))}{h} \\
&= \frac{1}{h}\int_0^1 \frac{d}{ds}\left(F_{p_i}\left(sDu(x+he_k) + (1-s)Du(x)\right)\right) ds \\
&= \int_0^1 F_{p_i p_j}\left((sDu(x+he_k) + (1-s)Du(x)\right) ds \frac{(D_j u(x+he_k) - D_j(x))}{h} \\
&= \sum_{j=1}^{n} a_{ij}^h(x) \Delta^h(D_j u),
\end{aligned}$$

where

$$a_{ij}^h(x) := \int_0^1 F_{p_i p_j}\left(sDu(x+he_k) + (1-s)Du(x)\right) ds.$$

Thus we obtain

$$\begin{aligned}
\int_\Omega \Delta^h \left(F_{p_i}(Du)\right) D_i(\varphi^2 \Delta^h u) &= \int_\Omega a_{ij}^h \Delta^h(D_j u) D_i(\varphi^2 \Delta^h u) \\
&=: A + B,
\end{aligned}$$

where

$$\begin{aligned}
A &:= \int_\Omega \varphi^2 a_{ij}^h \Delta^h(D_j u)\Delta^h(D_i u), \\
B &:= \int_\Omega 2a_{ij}^h \Delta^h(D_j u)\left(\Delta^h u\right)\varphi D_i\varphi.
\end{aligned}$$

Uniform convexity implies that

$$A \geq \theta \int_\Omega \varphi^2 |\Delta^h(Du)|^2,$$

and we can estimate B as follows:

$$|B| \leq C \int_{\mathcal{U}} \varphi |\Delta^h(Du)||\Delta^h u|,$$

since $|D_p^2 F(Du)| \leq C$ implies that $|a_{ij}^h(x)| \leq C$, note also we are now integrating over $\mathcal{U}$, and then

$$C \int_{\mathcal{U}} \varphi |\Delta^h(Du)||\Delta^h u| \leq \epsilon \int_\Omega \varphi^2 |\Delta^h(Du)|^2 + \frac{C}{\epsilon} \int_{\Omega'} |\Delta^h u|^2.$$

Thus we obtain a bound

$$\begin{aligned} \int_\Omega \varphi^2 |\Delta^h(Du)|^2 &\leq C \int_{\mathcal{U}} |\Delta^h u|^2 \\ &\leq C' \|Du\|_{L^2(\Omega)}, \end{aligned}$$

by the first difference quotient lemma (Proposition 6.11). But then as $\varphi \equiv 1$ on $\mathcal{O}$, we obtain

$$\left\| \Delta_k^h(Du) \right\|_{L^2(\mathcal{O})} \leq C \|Du\|_{L^2(\Omega)}$$

for each $k = 1, \ldots, n$ and $|h|$ sufficiently small. The second difference quotient lemma (Proposition 6.13) then shows that $Du \in W^{1,2}(\mathcal{O})$, and hence $u \in W^{2,2}(\mathcal{O})$. This is true for each $\mathcal{O} \subset\subset \Omega$, and so we conclude that $u \in W^{2,2}_{\text{loc}}(\Omega)$ as desired. ▶

9.10 Higher regularity

Unfortunately we cannot simply use induction to get higher order weak derivatives (and thus eventually, smoothness) as in the linear setting of Chapter 8. Indeed, given $\varphi \in C_c^\infty(\Omega)$, inserting $D_k\varphi$ into the equation gives

$$\int_\Omega F_{p_j}(Du) D_i(D_k \varphi) = 0,$$

and setting $w = D_k u \in W^{1,2}(\Omega)$ we obtain by integration by parts that

$$\int_\Omega F_{p_i p_j}(Du) D_j w D_j \varphi = 0,$$

and then by approximation this holds for any $v \in W_0^{1,2}(\Omega)$, and thus w weakly solves the equation

$$D_i(F_{p_i p_j}(Du) D_j w) = 0 \text{ in } \Omega.$$

Setting $a_{ij} := F_{p_i p_j}(Du)$ we have w a weak solution of

$$D_i(a_{ij} D_j w) = 0 \text{ in } \Omega.$$

Unfortunately, we **cannot** apply the regularity theory from Chapter 8 (eg. Theorem 8.1), since we only know that $a_{ij} \in L^\infty(\Omega)$. To get any further we need the following deep results:

9.11 Theorem (De-Giorgi, Nash, Moser)

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. Suppose $a_{ij} \in L^\infty(\Omega)$ and the equation

$$Lu = D_i(a_{ij} D_j u) = 0$$

is uniformly elliptic in Ω with constant of uniform ellipiticity λ. Suppose $u \in W^{1,2}(\Omega)$ is a weak solution. Then u is locally Hölder continuous: there exists $\alpha \in (0,1)$, where $\alpha = \alpha\left(n, \lambda, \|a_{ij}\|_{L^\infty(\Omega)}\right)$ such that $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$. Moreover, we have the following local estimate: for any $\mathcal{O} \subset\subset \Omega$,

$$\|u\|_{C^{0,\alpha}(\mathcal{O})} \leq C \|u\|_{L^p(\Omega)}$$

where $C = C\left(n, \lambda, \|a_{ij}\|_{L^\infty(\Omega)}, p, d\right)$, for $d = \text{dist}(\mathcal{O}, \partial\Omega)$.

9.12 Theorem (Schauder)

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. Suppose $a_{ij} \in C^{k,\gamma}_{\mathrm{loc}}(\Omega)$ and the equation

$$Lu = D_i(a_{ij}D_j u) = 0$$

is uniformly elliptic in Ω with constant of uniform ellipiticity λ. Suppose $u \in W^{1,2}(\Omega)$ is a weak solution. Then actually $u \in C^{k+1,\gamma}_{\mathrm{loc}}(\Omega)$.

Both Theorem 9.11 and Theorem 9.12 are in fact just special cases of more general results.

9.13 Elliptic bootstrapping

We can now deduce the desired regularity properties. Assume F is at least C^2. Then by Theorem 9.11 we deduce that $w \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$, and thus $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$. But then by definition of the a_{ij}, we then have $a_{ij} \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$, and thus Theorem 9.12 implies that $w \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$. But then $u \in C^{2,\alpha}_{\mathrm{loc}}(\Omega)$. Now repeat: we can keep doing this '**bootstrapping**' method until we are prevented from going any further by insufficient regularity of F.

In particular, if F is smooth, satisfies the conditions in Section 9.8 and $u \in W^{1,2}(\Omega)$ is a weak solution of

$$D_i\left(F_{p_i}(Du)\right) = 0 \text{ in } \Omega,$$

then in fact u is smooth. To sum up, we have proved:

9.14 Theorem

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary. Let F be a smooth Lagrangian satisfying the conditions of Section 9.8. Then there exists a smooth minimizer of the variational functional ($\spadesuit$), which in turn is a smooth solution of the Euler-Lagrange equations

$$D_i\left(F_{p_i}(Du)\right) = 0 \text{ in } \Omega.$$